COMPLEX VARIABLES: MID TERM 1

- The use of notes, calculators etc. is not permitted.
- There is a formula sheet provided on the last page for your use.
- Do not forget to write your name and UNI in the space provided below.

Name: ____________________________

UNI: ____________________________

For grader's use only.

1	**2**	**3**	**4**	**5**	Total
___ / 10	___ / 10	___ / 10	___ / 10	___ / 10	___ / 50

Problem 1 (10 points) Compute $\int_C \frac{3z^3+2}{z^2-1}\, dz$ where C is the counterclockwise oriented circle of radius 2 centered at 0.

Problem 2 (10 points) $\int_C \frac{1}{z^3 - z^2 + z - 1}\, dz$ where C is the counterclockwise oriented ellipse with equation $4x^2 + \frac{y^2}{4} = 1$.

Problem 3 (10 points) Let γ be the following simple curve, joining 1 and 5:

$$\gamma(t) = \begin{cases} \cos(t) + i\sin(t) & 0 \leq t \leq \pi \\ 2 + 3(\cos(t) + i\sin(t)) & \pi \leq t \leq 2\pi \end{cases}$$

Prove that

$$\int_\gamma \frac{1}{z}\, dz = \ln(5) + 2\pi i$$

Problem 4 (10 points) Consider the following function, where $x = \text{Re}(z)$ and $y = \text{Im}(z)$.

$$f(z) = (e^x + e^{-x})\cos(y) + i(e^x - e^{-x})\sin(y)$$

Verify that $f(z)$ and $f'(z)$ are $\mathbb{C}$-differentiable. Prove that $f''(z) = f(z)$.

Problem 5 (10 points) Let $f(z) = \dfrac{1}{z-1}$. Prove that for z lying on the vertical line $\mathrm{Re}(z) = 2$, the values of $f(z)$ lie on a circle. Find the radius and center of this circle.

Formula sheet

Statements of important results.

(1) $f(z) = u(x,y) + iv(x,y)$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Then f is $\mathbb{C}$–differentiable if, and only if the partial derivatives of u and v exist and are continuous, and $u_x = v_y$ and $u_y = -v_x$. In this case, $f'(z) = u_x + iv_x$.

(2) **Cauchy's Theorem.** If $f : D \to \mathbb{C}$ is $\mathbb{C}$–differentiable, where $D \subset \mathbb{C}$ is a non–empty open set, and γ is a simple closed curve whose interior is contained in D, then $\int_\gamma f(z)\,dz = 0$.

(3) **Cauchy's Integral formula.** With the same hypotheses on f, D, γ as in the previous statement, assume $a \in \mathrm{Int}(\gamma)$, then $\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$.

(4) Let $p(z), q(z)$ be two polynomials of degree n and m respectively. Let γ be a simple closed curve, which contains all the roots of $q(z)$ in its interior. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{p(z)}{q(z)}\,dz = \begin{cases} 0 & \text{if} \quad m \geq n+2 \\ \frac{a_n}{b_{n+1}} & \text{if} \quad m = n+1 \end{cases}$$

where a_n, b_m are coefficients of z^n, z^m in $p(z)$ and $q(z)$ respectively.

Anti-derivatives of a few functions over $\mathbb{R}$.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \ (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln(x)$$

$$\int \sin(x)\,dx = -\cos(x) \qquad \int \cos(x)\,dx = \sin(x)$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) \qquad \int \sec^2(x)\,dx = \tan(x)$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)$$

Trigonometric identities.

$$\sin^2(x) + \cos^2(x) = 1$$
$$\sec^2(x) - \tan^2(x) = 1$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$