Solutions to Practice Mid Term II

(1) $\sum_{n=2}^{\infty} \sin\left(\frac{\pi}{n}\right)$ is not convergent. Since for $0 \leq \theta \leq \frac{\pi}{2}$, $\sin(\theta) \geq \frac{2}{\pi}\theta$

(Proof We proved in class that $\frac{\sin(\theta)}{\theta}$ is decreasing function for $0 \leq \theta \leq \frac{\pi}{2}$, hence $\frac{\sin(\theta)}{\theta} \geq \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi}$)

Therefore $\sin\left(\frac{\pi}{n}\right) \geq \frac{2}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{n}$ for $n \geq 2$

$$= \frac{2}{n}$$

$\cancel{\Rightarrow}$ But $\sum_{n=2}^{\infty} \frac{2}{n}$ is divergent. $\left(\text{Because } \sum_{n=2}^{\infty} \frac{1}{n} \geq \int_{2}^{\infty} \frac{1}{x}\, dx = \infty\right)$

$\Rightarrow$ $\sum_{n=2}^{\infty} \sin\left(\frac{\pi}{n}\right)$ is also divergent. $\square$

(2) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$. Ratio of successive terms

$$= \frac{(n+1)!}{(n+1)^{n+1}} z^{n+1} \frac{n^n}{n!} \frac{1}{z^n} = \frac{n+1}{(n+1)^{n+1}} n^n z$$

$$= \frac{n^n}{(n+1)^n} z = \frac{1}{\left(1+\frac{1}{n}\right)^n} z$$

So $\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} |z| = \frac{|z|}{e} < 1$ for $|z| < e$

$\Rightarrow$ Radius of convergence $= e$.

(3) Since $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$

and power series can be differentiated termwise. We get

$$\frac{d}{dz} \sum_{n=0}^{\infty} z^n = \frac{d}{dz}\left(\frac{1}{1-z}\right) \Rightarrow \sum_{n=1}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} n z^n = \frac{z}{(1-z)^2}$$

Differentiate again: $\sum_{n=1}^{\infty} n^2 z^{n-1} = \frac{d}{dz} \frac{z}{(1-z)^2} = \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3}$

$$= \frac{1+z}{(1-z)^3} \qquad (|z| < 1)$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 z^n = \frac{z(1+z)}{(1-z)^3} \quad \text{for } |z| < 1$$

(4) $\operatorname{Res}_{z=0} \frac{1}{z^4 (1-z)(2-z)}$ = Coeff. of z^3 in the Taylor Series expansion of $\frac{1}{1-z} \cdot \frac{1}{2-z}$ near $z = 0$

$$= \text{Coefficient of } z^3 \text{ in } \left(1 + z + z^2 + z^3 + \cdots\right) \cdot \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$$

$$= \frac{1}{2}\left(\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1\right) = \frac{1}{2} \, \frac{1+2+4+8}{8} = \frac{15}{16}.$$

$$\left(\text{since } \frac{1}{2-z} = \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n} \text{ for } |z| < 2\right)$$

(5) $\displaystyle\int_0^{2\pi} \frac{1}{2+\cos(\theta)}\, d\theta$. Set $z = e^{i\theta}$ $dz = iz\, d\theta$ ③

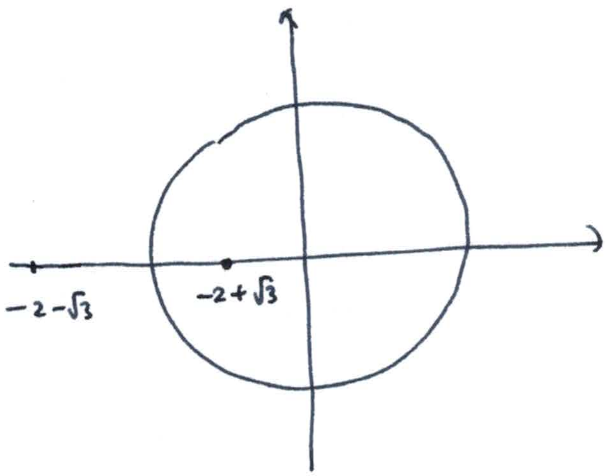
$\cos(\theta) = \dfrac{z+z^{-1}}{2}$

$$= \int_\gamma \frac{1}{2+\frac{z+z^{-1}}{2}} \frac{dz}{iz} = \frac{2}{i}\int_\gamma \frac{1}{z^2+4z+1}\, dz$$

(γ = circle of radius 1 centered at 0)

$$\boxed{\begin{aligned} z^2+4z+1=0 \Rightarrow z &= \frac{-4\pm\sqrt{12}}{2} \\ &= -2\pm\sqrt{3} \end{aligned}}$$

$$= \frac{2}{i}\int_\gamma \frac{1}{(z+2-\sqrt{3})(z+2+\sqrt{3})}\, dz$$



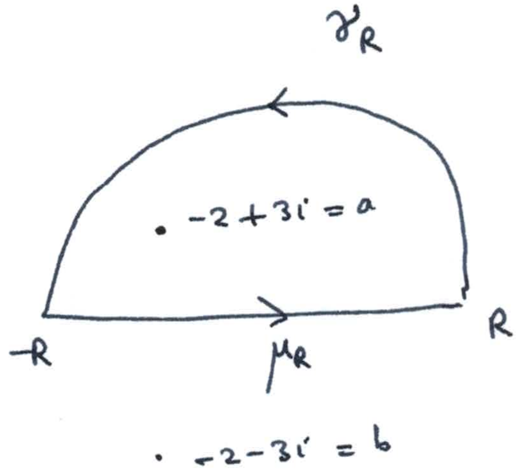
$$= \frac{2}{i}\cdot 2\pi i \left.\frac{1}{z+2+\sqrt{3}}\right|_{\text{set } z=-2+\sqrt{3}}$$

(by Cauchy's formula)

$$= 4\pi \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

(6) $\displaystyle \lim_{R \to \infty} \int_{-R}^{R} \frac{x}{(x^2+4x+13)^2}\, dx$

$C_R = \gamma_R + \mu_R$:



$$z^2 + 4z + 13 = 0$$

$$\Rightarrow \quad z = \frac{-4 \pm \sqrt{16-52}}{2}$$

$$= -2 \pm 3i$$

$$\int_{C_R} \frac{z}{(z^2+4z+13)^2}\, dz = \int_{C_R} \frac{z}{(z-a)^2(z-b)^2}\, dz = 2\pi i \left[\frac{d}{dz} \frac{z}{(z-b)^2}\right] \text{ set } z=a$$

(Cauchy's formula)

$$= 2\pi i \left[\frac{1}{(z-b)^2} - \frac{2z}{(z-b)^3}\right]_{z=a}$$

$$= 2\pi i \left[\frac{z-b-2z}{(z-b)^3}\right]_{z=a} = -2\pi i \frac{a+b}{(a-b)^3}$$

$$= -2\pi i \frac{(-4)}{(6i)^3}$$

$$= \frac{8\pi i}{-216 i} = -\frac{\pi}{27}$$

Now $\displaystyle \left| \int_{\gamma_R} \frac{z}{(z-a)^2(z-b)^2}\, dz \right| \leq \frac{R}{(R-|a|)^2(R-|b|)^2} \cdot \pi R \to 0$ as $R \to \infty$.

Hence $\boxed{\displaystyle \lim_{R\to\infty} \int_{-R}^{R} \frac{x}{(x^2+4x+13)^2}\, dx = \frac{-\pi}{27}}$