COMPLEX VARIABLES : MID TERM II

Instructions:

- For full credit, please show all your work.
- There is a formula sheet provided at the end.
- The use of any other material (calculators, class notes etc.) is not permitted.
- Do not forget to write your name and UNI at the space provided below.

Name: Solutions
UNI: ______

For graders use only.

1	2	3	4	5	Total
__/10	__/10	__/10	__/10	__/10	__/50

Average 41.1

Median 41.5

Std. Dev. 6.03

Problem 1. (10 points) Write the Taylor series expansion of $\frac{1}{z^2 - z + 1}$ around $z = 0$.

$$z^2 - z + 1 = (z-\alpha)(z-\beta) \quad \text{when} \quad \alpha/\beta = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\frac{1}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha-\beta}\left(\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right)$$

$$= \frac{1}{\alpha-\beta}\left(-\sum_{n=0}^{\infty} \frac{z^n}{\alpha^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{\beta^{n+1}}\right)$$

$$= \sum_{n=0}^{\infty} z^n \left(\frac{\beta^{-n-1} - \alpha^{-n-1}}{\alpha-\beta}\right)$$

Alternately, $\frac{1}{z^2 - z + 1} = \frac{z+1}{(z+1)(z^2-z+1)} = \frac{z+1}{z^3+1}$

$$= (z+1) \sum_{n=0}^{\infty} (-1)^n z^{3n}$$

$$= \sum_{n=0}^{\infty} \left(z^{3n+1} + z^{3n}\right)(-1)^n$$

$$= 1 + z - z^3 - z^4 + z^6 + z^7 - z^9 - z^{10} \cdots$$

Problem 2. (10 points) What is its radius of convergence of $\sum_{n=0}^{\infty} \sin\left(n\frac{\pi}{3}\right) z^n$?

(Hint: write a first few terms of the series)

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{3\pi}{3}\right) = 0$$

$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \qquad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2} \qquad \sin\left(\frac{6\pi}{3}\right) = 0$$

...

$$\sum_{n=0}^{\infty} \sin\left(n\frac{\pi}{3}\right) z^n = \frac{\sqrt{3}}{2} z + \frac{\sqrt{3}}{2} z^2 + 0 \cdot z^3 - \frac{\sqrt{3}}{2} z^4 - \frac{\sqrt{3}}{2} z^5 + 0 \cdot z^6 + \frac{\sqrt{3}}{2} z^7 + \frac{\sqrt{3}}{2} z^8 + 0 \cdot z^9 - \frac{\sqrt{3}}{2} z^{10} - \frac{\sqrt{3}}{2} z^{11} + 0 \cdot z^{12} \ldots$$

$$\left| \begin{array}{l} \text{ratio of consecutive} \\ \text{non-zero terms} \end{array} \right| = |z| \text{ or } |z|^2 \implies \text{Radius of convergence} = 1$$

Alternately, $\displaystyle\sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{3}\right) z^n = \frac{\sqrt{3}}{2}\left(z + z^2 - z^4 - z^5\right)\left(\sum_{n=0}^{\infty} z^{6n}\right)$

$\uparrow$ has radius of convergence $= 1$.

Problem 3. (10 points) Compute the following residue

$$\operatorname{Res}_{z=0}\left(\frac{\log(1+z)\sin(z)}{z^5}\right)$$

= Coefficient of z^4 in $\log(1+z)\sin(z)$'s Taylor series expansion around 0.

= Coefficient of z^4 in

$$\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots\right)\cdot\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= -\frac{1}{3!} + \frac{1}{3} = -\frac{1}{6} + \frac{1}{3} = \frac{1}{6}.$$

Common Mistakes: 3 points off if • $z = e^{ix}$ and did not justify switch from semicircle to full circle

or • used log without specifying which log.

Problem 4. (10 points) Prove that for any $a > 0$ real number, we have

$$\int_0^\pi \tan(x + ai)\,dx = \pi i$$

$$\tan(x+ai) = \frac{\sin(x+ai)}{\cos(x+ai)} = \frac{e^{i(x+ai)} - e^{-i(x+ai)}}{i\left(e^{i(x+ai)} + e^{i(x+ai)}\right)}$$

$$= \frac{e^{ix}e^{-a} - e^{-ix}e^{a}}{i\left(e^{ix}e^{-a} + e^{-ix}e^{a}\right)} = \frac{e^{2ix} - e^{2a}}{i\left(e^{2ix} + e^{2a}\right)}$$

Set $z = e^{2ix}$ $\quad dz = 2iz\,dx$

$$\int_0^\pi \tan(x+ai)\,dx = \int_\gamma \frac{z - e^{2a}}{i(z+e^{2a})} \frac{dz}{2iz}$$

$\gamma =$ circle of radius 1 centered at 0

$$= -\frac{1}{2} \int_\gamma \frac{z - e^{2a}}{(z+e^{2a})z}\,dz$$

$-e^{2a} < -1$ is outside of γ

$$= -\frac{1}{2}\, 2\pi i \left[\frac{z - e^{2a}}{z + e^{2a}}\right]_{\text{set } z=0}$$

$$= -\pi i\,(-1) = \pi i \qquad \square$$

4 points off if applied Jordan's Lemma to $\frac{z\sin(z)}{z^2+9}$.

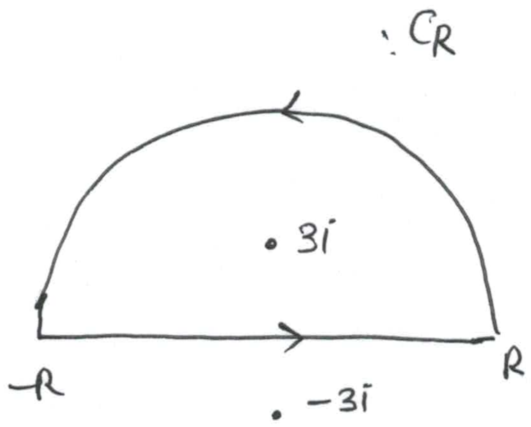
Problem 5. (10 points) Compute the integral $\int_0^\infty \frac{x\sin(x)}{x^2+9}\,dx$.

$\left(\text{Hint: Jordan's lemma and } \operatorname{Im}\left(\frac{ze^{iz}}{z^2+9}\right) = \frac{z\sin(z)}{z^2+9} \text{ for } z \in \mathbb{R}\right)$

$$\int_{C_R} \frac{ze^{iz}}{z^2+9}\,dz = 2\pi i \left[\frac{ze^{iz}}{z+3i}\right]_{z=3i}$$

$$= \frac{2\pi i}{6i}\, 3i\, e^{i(3i)}$$

$$= \pi i e^{-3}$$



$\gamma_R(\theta) = Re^{i\theta} \quad 0 \le \theta \le \pi$

By Jordan's Lemma

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{ze^{iz}}{z^2+9}\,dz = 0$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{xe^{ix}}{x^2+9}\,dx = \pi e^{-3} i$$

Take im. part:

$$2 \lim_{R\to\infty} \int_0^R \frac{x\sin(x)}{x^2+9}\,dx = \pi e^{-3}$$

$$\Rightarrow \int_0^\infty \frac{x\sin(x)}{x^2+9}\,dx = \frac{\pi}{2} e^{-3}.$$