COMPLEX VARIABLES: HOMEWORK 1

Notations: A complex number z is written as $x + yi$ where x, y are real numbers. Its real and imaginar parts are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively.

(1) **(5 points)** Find the real and imaginary parts of the $\dfrac{1-i}{2+3i}$.

Solution. Multiply and divide by the conjugate of the denominator.

$$\frac{1-i}{2+3i} = \frac{1-i}{2+3i}\frac{2-3i}{2-3i} = \frac{-1-5i}{13}$$

Hence the real part is $\frac{-1}{13}$ and the imaginary part is $\frac{-5}{13}$.

(2) **(5 points)** Write $z = -1 + i$ in its polar form. Use this to compute z^{20}.

Solution. $|z| = \sqrt{2}$ and $\arg(z) = \frac{3\pi}{4}$. Thus

$$z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right) i\right)$$

So, $|z^{20}| = (\sqrt{2})^{20} = 2^{10} = 1024$. And $\arg(z^{20}) = 20\frac{3\pi}{4} = 15\pi$. Hence we get

$$z^{20} = 1024\left(\cos(15\pi) + \sin(15\pi)i\right) = -1024$$

(3) **(10 points) (2+3+5)** Recall that the conjugate of $z = x + yi$ is defined as $\overline{z} = x - yi$. Prove the following, for any two complex number z_1 and z_2.

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Proof. Let us write $z_1 = a + bi$ and $z_2 = c + di$. Then

$$\overline{z_1 + z_2} = (a + c) - (b + d)i = (a - bi) + (c - di) = \overline{z_1} + \overline{z_2}$$

as required.

(b) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

Proof. Again we can compute the left–hand side as

$$\overline{z_1 z_2} = \overline{ac - bd + (ad + bc)i} = (ac - bd) - (ad + bc)i = (a - bi)(c - di) = \overline{z_1}\ \ \overline{z_2}$$

(c) $|z_1 + z_2|^2 = |z_1|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2$.

Proof. Let us start by writing $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$. (This is because, for any complex number w, we know $|w|^2 = w\overline{w}$). Hence the left–hand side can be written as

$$\begin{aligned} \text{L.H.S.} &= z_1\overline{z_1} + z_2\overline{z_2} + (z_1\overline{z_2} + \overline{z_1}z_2) \\ &= |z_1|^2 + |z_2|^2 + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) \\ &= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) \end{aligned}$$

(since for any complex number w, we know that $w + \overline{w} = 2\,\mathrm{Re}(w)$).

(4) **(15 points)=5+5+5** Let $z_1, \cdots, z_n$ be n distinct solutions of the equation $z^n = 1$.

This means that

$$(*) \qquad z^n - 1 = (z - z_1)(z - z_2) \cdots (z - z_n)$$

(a) Product of $z_1, \cdots, z_n$ is $(-1)^{n-1}$.

Proof. Take the constant term on both sides of equation $(*)$. The left–hand side gives -1 and the right–hand side gives $(-1)^n$ times product of all $z_1, \cdots, z_n$. Hence, we get

$$(-1)^n(z_1 z_2 \cdots z_n) = -1 \Rightarrow (z_1 z_2 \cdots z_n) = (-1)^{n-1}$$

(b) Sum of $z_1, \cdots, z_n$ is 0.

Proof. Take the coefficient of z^1 on both sides of equation $(*)$. The left–hand side gives 0 (since $n \geq 2$) and the right–hand side gives the sum of all $z_1, \cdots, z_n$.

(c) If $z_1 = 1$, then $(1 - z_2)(1 - z_3) \cdots (1 - z_n) = n$.

Proof. Let us divide both sides of equation $(*)$ by $z - 1$:

$$\frac{z^n - 1}{z - 1} = (z - z_2)(z - z_3) \cdots (z - z_n)$$

Now use the following identity

$$\frac{z^n - 1}{z - 1} = z^{n-1} + z^{n-2} + \cdots + z + 1$$

Then we get

$$(z - z_2)(z - z_3) \cdots (z - z_n) = z^{n-1} + z^{n-2} + \cdots + z + 1$$

Set $z = 1$ in this equation, to get

$$(1 - z_2)(1 - z_3) \cdots (1 - z_n) = n$$

(5) **(5 points)** Let $f : D \to \mathbb{C}$ be a function, where $D \subset \mathbb{C}$ is an open set. Let $u(x, y)$ and $v(x, y)$ be its real and imaginary parts respectively:

$$f(x + yi) = u(x, y) + v(x, y)i$$

(Assume that partial derivatives of $u(x, y)$ and $v(x, y)$ exist and are continuous). Prove that, if f is $\mathbb{C}$–differentiable, then $u_{xx} + u_{yy} = 0$. Use this to show that there is no $\mathbb{C}$–differentiable function whose real part is e^x.

Proof. By Cauchy–Riemann equations, we have $u_x = v_y$ and $u_y = -v_x$. Hence

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

Since for $u(x, y) = e^x$, we get $u_{xx} + u_{yy} = e^x + 0 \neq 0$, there cannot be any $\mathbb{C}$–differentiable function f whose real part is e^x.

(6) **(10 points)** Prove that for any two complex numbers z_1, z_2 the following inequality holds

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

Prove that this inequality is an equality if, and only if $\arg(z_1) = \arg(z_2)$ (or one of them is zero).

Proof. Use (c) of Problem (3) above to see that

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1 \overline{z_2})$$

Now $\arg(z_1\overline{z_2}) = \arg(z_1) + \arg(\overline{z_2}) = \arg(z_1) - \arg(z_2)$ (if neither of them is zero). Note that if one of z_1 or z_2 is zero, then the statement to prove is obviously true. Moreover $|z_1\overline{z_2}| = |z_1||\overline{z_2}| = |z_1||z_2|$. Hence we get

$$\begin{aligned}
|z_1 - z_2|^2 &= |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\overline{z_2}) \\
&= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\arg(z_1) - \arg(z_2)) \\
&\geq |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \\
&= (|z_1| - |z_2|)^2
\end{aligned}$$

Thus taking square–root gives $|z_1 - z_2| \geq ||z_1| - |z_2||$. Also, the inequality in the third line above is equality, if and only if $\cos(\arg(z_1) - \arg(z_2)) = 1$, that is, $\arg(z_1) = \arg(z_2)$.