COMPLEX VARIABLES: HOMEWORK 1

Notations: A complex number z is written as $x + yi$ where x, y are real numbers. Its real and imaginar parts are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively.

(1) Find the real and imaginary parts of the $\dfrac{1-i}{2+3i}$.

(2) Write $z = -1 + i$ in its polar form. Use this to compute z^{20}.

(3) Recall that the conjugate of $z = x + yi$ is defined as $\overline{z} = x - yi$. Prove the following, for any two complex number z_1 and z_2.
   (a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

   (b) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

   (c) $|z_1 + z_2|^2 = |z_1|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2$.

(4) Let $z_1, \cdots, z_n$ be n distinct solutions of the equation $z^n = 1$ (assume $n \geq 2$). Prove that
   (a) Product of $z_1, \cdots, z_n$ is $(-1)^{n-1}$.

   (b) The sum of $z_1, \cdots, z_n$ is 0.

   (c) If $z_1 = 1$, then $(1 - z_2)(1 - z_3) \cdots (1 - z_n) = n$.

(5) Let $f : D \to \mathbb{C}$ be a function, where $D \subset \mathbb{C}$ is an open set. Let $u(x, y)$ and $v(x, y)$ be its real and imaginary parts respectively:

$$f(x + yi) = u(x, y) + v(x, y)i$$

(Assume that partial derivatives of $u(x, y)$ and $v(x, y)$ exist and are continuous). Prove that, if f is $\mathbb{C}$–differentiable, then $u_{xx} + u_{yy} = 0$. Use this to show that there is no $\mathbb{C}$–differentiable function whose real part is e^x.

(6) Prove that for any two complex numbers z_1, z_2 the following inequality holds

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

Prove that this inequality is an equality if, and only if $\arg(z_1) = \arg(z_2)$ (or one of them is zero).