COMPLEX VARIABLES: HOMEWORK 10

Recall that we have the following set up. $\tau \in \mathbb{C}$ is a complex number lying in the upper half plane, that is, $\mathrm{Im}(\tau) > 0$. Let $q = e^{\pi i \tau}$ and let $\theta(z;\tau)$ be the holomorphic function defined as:

$$\theta(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} e^{2\pi i n z}$$

(1) Prove the following

$$\theta(z;\tau) = 2ie^{\pi i z}\left(\sum_{n=1}^{\infty} (-1)^n q^{n(n-1)} \sin((2n-1)\pi z)\right)$$

Solution. From the definition of $\theta(z;\tau)$ collect the terms which give same exponent of q: namely,

$$q^{n(n-1)} = q^{-n(-n+1)} = q^{m(m-1)} \text{ where } m = -n+1$$

Therefore we get

$$\begin{aligned}\theta(z;\tau) &= \sum_{n\geq 0} (-1)^n q^{n(n-1)} \left(e^{2\pi i n z} - e^{2\pi i(-n+1)z}\right)\\ &= e^{\pi i z}\sum_{n\geq 0} (-1)^n q^{n(n-1)} \left(e^{\pi i(2n-1)z} - e^{-\pi i(2n-1)z}\right)\\ &= 2ie^{\pi i z}\sum_{n\geq 0} (-1)^n q^{n(n-1)} \sin((2n-1)\pi z)\end{aligned}$$

as required.

(2) What is the limit of $\theta(z;\tau)$ as the imaginary part of τ goes to infinity? That is, compute the following:

$$\lim_{\mathrm{Im}(\tau)\to\infty} \theta(z;\tau)$$

Solution. After setting $q = 0$ the only remaining terms in the formula of $\theta(z;\tau)$ are the ones corresponding to $n = 0, 1$. Consequently, we get

$$\lim_{\mathrm{Im}(\tau)\to\infty} \theta(z;\tau) = 1 - e^{2\pi i z}$$

(3) Consider the system of equations for an unknown function $f(z)$:

$$f(z+1) = f(z) \quad \text{and} \quad f(z+\tau) = e^{2\pi i a} f(z)$$

where $a \in \mathbb{C}$ is a complex number.

(a) Use the theta function to write a solution of these equations.

Solution. $\dfrac{\theta(z-a;\tau)}{\theta(z;\tau)}$.

(b) Prove that if $f_1(z)$ and $f_2(z)$ are two solutions, then their ratio is an elliptic function.

Solution. Since both f_1 and f_2 satisfy the system of equations given above, we get:

$$\frac{f_1(z+1)}{f_2(z+1)} = \frac{f_1(z)}{f_2(z)} \text{ and } \frac{f_1(z+\tau)}{f_2(z+\tau)} = \frac{e^{2\pi i a} f_1(z)}{e^{2\pi i a} f_2(z)} = \frac{f_1(z)}{f_2(z)}$$

Hence the ratio f_1/f_2 is elliptic.

(c) Combine the previous two parts to prove the following: assuming $a \neq m + n\tau$ for any $m, n \in \mathbb{Z}$, there are no holomorphic solutions to these equations: ($f(z+1) = f(z)$ and $f(z+\tau) = e^{2\pi i a} f(z)$).

Solution. Assume that $f(z)$ is a holomorphic solution. Then from the previous two parts we see that $f(z)\dfrac{\theta(z;\tau)}{\theta(z-a;\tau)}$ is an elliptic function with at most one simple pole in any fundamental parallelogram. Thus it must be a constant (i.e, no poles at all). But that means

$$f(z) = \text{Constant.}\frac{\theta(z-a;\tau)}{\theta(z;\tau)}$$

is holomorphic function. For the ratio of theta functions on the right–hand side, this is only possible when $a = m + n\tau$ for some integers $m, n \in \mathbb{Z}$.

(4) Recall that $\theta_2(z;\tau) = \theta\left(z + \frac{1}{2}; \tau\right)$. Carry out the computations given in sections (20.5) and (20.6) of Lecture 20, for θ_2 to prove the following:

$$\frac{1}{\pi i}\left(\frac{1}{\theta_2(0;\tau)}\frac{\partial}{\partial \tau}\theta_2(0;\tau)\right) = 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2}$$

Solution. We use the heat equation:

$$\frac{1}{\pi i}\partial_\tau = \frac{1}{(2\pi i)^2}\partial_z^2 - \frac{1}{2\pi i}\partial_z$$

to reduce the problem to computing

$$\frac{1}{\pi i}\left(\frac{1}{\theta_2(0;\tau)}\frac{\partial}{\partial \tau}\theta_2(0;\tau)\right) = \frac{1}{(2\pi i)^2}\frac{\theta_2''(0)}{\theta_2(0)} - \frac{1}{2\pi i}\frac{\theta_2'(0)}{\theta_2(0)}$$

Now $\theta_2(z;\tau) = G\prod_{n\geq 0}(1+q^{2n}e^{2\pi i z})\prod_{n\geq 1}(1+q^{2n}e^{-2\pi i z})$. So taking logarithmic derivative gives:

$$\frac{\theta_2'(z)}{\theta_2(z)} = \sum_{n\geq 0}\frac{q^{2n}2\pi i e^{2\pi i z}}{1+q^{2n}e^{2\pi i z}} + \sum_{n\geq 0}\frac{q^{2n}(-2\pi i)e^{-2\pi i z}}{1+q^{2n}e^{-2\pi i z}}$$

Take derivative again to get:

$$\frac{\theta_2''(z)}{\theta_2(z)} - \left(\frac{\theta_2'(z)}{\theta_2(z)}\right)^2 = \sum_{n\geq 0}\frac{q^{2n}(2\pi i)^2 e^{2\pi i z}}{(1+q^{2n}e^{2\pi i z})^2} + \sum_{n\geq 0}\frac{q^{2n}(-2\pi i)^2 e^{-2\pi i z}}{(1+q^{2n}e^{-2\pi i z})^2}$$

Setting $z = 0$ in these two equations implies:

$$\frac{\theta_2'(0)}{\theta_2(0)} = \pi i$$

$$\frac{\theta_2''(0)}{\theta_2(0)} = 2(\pi i)^2 + 2\sum_{n\geq 1}\frac{q^{2n}(2\pi i)^2}{(1+q^{2n})^2}$$

Subsititute it back in the original equation

$$\frac{1}{(2\pi i)^2}\frac{\theta_2''(0)}{\theta_2(0)} - \frac{1}{2\pi i}\frac{\theta_2'(0)}{\theta_2(0)} = \frac{1}{2} + 2\sum_{n\geq 1}\frac{q^{2n}}{(1+q^{2n})^2} - \frac{1}{2}$$

to get the desired result.