COMPLEX VARIABLES: HOMEWORK 10

Recall that we have the following set up. $\tau \in \mathbb{C}$ is a complex number lying in the upper half plane, that is, $\mathrm{Im}(\tau) > 0$. Let $q = e^{\pi i \tau}$ and let $\theta(z;\tau)$ be the holomorphic function defined as:

$$\theta(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} e^{2\pi i n z}$$

(1) Prove the following

$$\theta(z;\tau) = 2ie^{\pi i z} \left(\sum_{n=1}^{\infty} (-1)^n q^{n(n-1)} \sin((2n-1)\pi z) \right)$$

(2) What is the limit of $\theta(z;\tau)$ as the imaginary part of τ goes to infinity? That is, compute the following:

$$\lim_{\mathrm{Im}(\tau)\to\infty} \theta(z;\tau)$$

(3) Consider the system of equations for an unknown function $f(z)$:

$$f(z+1) = f(z) \quad \text{and} \quad f(z+\tau) = e^{2\pi i a} f(z)$$

where $a \in \mathbb{C}$ is a complex number.

(a) Use the theta function to write a solution of these equations.

(b) Prove that if $f_1(z)$ and $f_2(z)$ are two solutions, then their ratio is an elliptic function.

(c) Combine the previous two parts to prove the following: assuming $a \neq m + n\tau$ for any $m, n \in \mathbb{Z}$, there are no holomorphic solutions to these equations: ($f(z+1) = f(z)$ and $f(z+\tau) = e^{2\pi i a} f(z)$).

(4) Recall that $\theta_2(z;\tau) = \theta\left(z + \frac{1}{2};\tau\right)$. Carry out the computations given in sections (20.5) and (20.6) of Lecture 20, for θ_2 to prove the following:

$$\frac{1}{\pi i}\left(\frac{1}{\theta_2(0;\tau)} \frac{\partial}{\partial\tau}\theta_2(0;\tau)\right) = 2\sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2}$$