COMPLEX VARIABLES: HOMEWORK 11

In the problems below, $\tau \in \mathbb{C}$ is such that $\mathrm{Im}(\tau) > 0$. Let $q = e^{\pi i \tau}$.

(1) Let $h \in \mathbb{C}$ be a non–zero complex number so that $ph \neq m + n\tau$ for any three integers $p, m, n \in \mathbb{Z}$, $p \neq 0$. Assume that $f(z)$ is a "triply–periodic" meromorphic function:

$$f(z+1) = f(z) \qquad f(z+\tau) = f(z) \qquad f(z+h) = f(z)$$

Prove that $f(z)$ has to be holomorphic and hence a constant.

Solution. Assume that $f(z)$ is not holomorphic and let $a \in P$ be a pole of $f(z)$, where P is a fundamental parallelogram. For every $w \in \mathbb{C}$ let us denote by $\widetilde{w} \in P$ the unique element in P which is congruent to w modulo Λ_τ. That is,

$$w - \widetilde{w} \in \Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$$

Then by the third periodicity requirement, $\widetilde{a + nh}$ is also a pole of $f(z)$ for every $n \in \mathbb{Z}$. Moreover, if $\widetilde{a + kh} = \widetilde{a + lh}$ for some $k \neq l$, then we will have

$$a + kh - a - lh = (k - l)h \in \Lambda_\tau$$

which is prohibited by given assumption on h. Therefore we get that if $f(z)$ has a pole in P then it has infinitely many poles in P contradicting its meromorphicity. Hence it has no poles and is therefore holomorphic.

(2) Recall that we defined $\theta_3(z;\tau) = \theta\left(z + \frac{1}{2} + \frac{\tau}{2}; \tau\right)$. Prove the following properties of $\theta_3(z;\tau)$.

(a) $\theta_3(z+1;\tau) = \theta_3(z;\tau)$ and $\theta_3(z+\tau;\tau) = q^{-1}e^{-2\pi i z}\theta_3(z;\tau)$.

Solution. The first equation is clear. For the second, we have

$$\theta_3(z+\tau) = \theta\left(\left(z + \frac{1+\tau}{2}\right) + \tau\right) = -e^{-2\pi i\left(z + \frac{1+\tau}{2}\right)}\theta\left(z + \frac{1+\tau}{2}\right) = q^{-1}e^{-2\pi i z}\theta_3(z)$$

as required.

(b) $\theta_3(-z;\tau) = \theta_3(z;\tau)$.

Solution. This follows clearly from the definition:

$$\theta_3(z) = \sum_{n \in \mathbb{Z}} q^{n^2} e^{2\pi i n z}$$

(c) $\theta_3(z;\tau)$ has zeroes with multiplicity one at $\frac{1+\tau}{2} + m + n\tau$ for every pair of integers $m, n \in \mathbb{Z}$.

Solution. $\theta_3(z) = 0$ if and only if $\theta(z + (1+\tau)/2) = 0$. For the latter we know that this is only possible for $z = (1+\tau)/2 + m + n\tau$ for some $m, n \in \mathbb{Z}$, each with multiplicity one.

(3) Consider the following function, for any three complex numbers $a, A, B \in \mathbb{C}$:

$$f(z) = e^{2\pi iz}\frac{A\theta_3(z-a)\theta_3(z+a) + B\theta_3(z)^2}{\theta(z)^2}$$

(a) Prove that $f(z)$ is elliptic.

Solution. $f(z+1) = f(z)$ is obvious. For the shift by τ, we have

$$\begin{aligned} f(z+\tau) &= q^2 e^{2\pi iz}\frac{q^{-2}e^{-4\pi iz}(A\theta_3(z-a)\theta_3(z+a) + B\theta_3(z)^2)}{e^{-4\pi iz}\theta_1(z)^2} \\ &= f(z) \end{aligned}$$

Hence $f(z)$ is elliptic.

(b) Find the values of A, B which make $f(z)$ holomorphic.

Solution. Since $f(z)$ has apparent pole of order 2 at $z = 0$ (in the fundamental parallelogram, say with corners $\pm(1/2) \pm (\tau/2)$), if we make sure that the numerator is also zero at $z = 0$, then this apparent pole would be atmost of order 1, and since for elliptic function we cannot have just one simple pole in a given funamental parallelogram, the resulting function would be holomorphic. Thus we need to find A, B so that

$$A\theta_3(-a)\theta_3(a) + B\theta_3(0)^2 = 0$$

One solution is provided by $A = \theta_3(0)^2$ and $B = -\theta_3(a)^2$.

(c) For these values of A, B the function $f(z)$ has to be constant. Determine the value of this constant.

Solution. Since for $A = \theta_3(0)^2$ and $B = -\theta_3(a)^2$ the function $f(z) = \text{Constant}$, we can determine this constant by setting $z = (1+\tau)/2$. Note that $\theta_3((1+\tau)/2) = 0$, so that we get:

$$\text{Constant } = e^{2\pi i(1+\tau)/2}\frac{\theta_3(0)^2\theta_3(-a+(1+\tau)/2)\theta_3(a+(1+\tau)/2)}{\theta((1+\tau)/2)^2}$$

Now $\theta((1+\tau)/2) = \theta_3(0)$. Also, we have

$$\theta_3(-a+(1+\tau)/2)\theta_3(a+(1+\tau)/2) = \theta(-a+1+\tau)\theta(a+1+\tau) = \theta(-a)\theta(a) = -e^{-2\pi ia}\theta(a)^2$$

Hence we get

$$\text{Constant } = -q(-e^{-2\pi ia})\theta(a)^2 = qe^{-2\pi ia}\theta(a)^2$$

(4) **Bonus.** The problem above gives a proof of the following identity (called addition formula for theta functions):

$$e^{2\pi iz}\left(\theta_3(0)^2\theta_3(z-a)\theta_3(z+a) - \theta_3(a)^2\theta_3(z)^2\right) = qe^{-2\pi ia}\theta(a)^2\theta(z)^2$$

Deduce this identity from Fay's trisecant identity proved in class.