COMPLEX VARIABLES: HOMEWORK 3 SOLUTIONS

All simple closed curves considered below are assumed to be counterclockwise oriented. You can use the results proved in class, except when it is explicitly prohibited (see problems 3 and 4).

(1) Prove that $\int_C z^{-n}\,dz$ equals 0 if $n > 1$ and $2\pi i$ if $n = 1$. Here n is a natural number and C is the counterclockwise circle of radius $R > 0$.

Solution. As an application of Cauchy's integral formula

$$\begin{aligned}\int_C \frac{1}{z^n}\,dz &= \frac{2\pi i}{(n-1)!}\left[(\frac{d}{dz})^{n-1}1\right]_{z=0} \\ &= 0 \text{ if } n \geq 2 \\ &= 2\pi i \text{ if } n = 1\end{aligned}$$

(2) Compute all possible values of $\int_C \frac{1}{z(z^2-1)}\,dz$ for different choices of simple closed curves C which do not pass through $0, 1, -1$.

Solution. $z(z^2-1) = 0$ if, and only if $z = 0, 1, -1$. There are following choices for C:

(a) C does not contain either $0, 1, -1$ in its interior:

$$\int_C \frac{1}{z(z-1)(z+1)}\,dz = 0$$

(b) C contains exactly one of them: let C_0, C_1, C_{-1} be the corresponding choices. Then by Cauchy's integral formula:

$$\int_{C_0} \frac{1}{z(z-1)(z+1)}\,dz = 2\pi i\frac{1}{(0-1)(0+1)} = -2\pi i$$
$$\int_{C_1} \frac{1}{z(z-1)(z+1)}\,dz = 2\pi i\frac{1}{1(1+1)} = \pi i$$
$$\int_{C_{-1}} \frac{1}{z(z-1)(z+1)}\,dz = 2\pi i\frac{1}{(-1)(-1-1)} = \pi i$$

(c) C contains two of $0, 1, -1$:

$$\int_{C_0+C_1} \frac{1}{z(z-1)(z+1)}\,dz = -2\pi i + \pi i = -\pi i$$
$$\int_{C_0+C_{-1}} \frac{1}{z(z-1)(z+1)}\,dz = -2\pi i + \pi i = -\pi i$$
$$\int_{C_1+C_{-1}} \frac{1}{z(z-1)(z+1)}\,dz = \pi i + \pi i = 2\pi i$$

(d) C contains all of them:

$$\int_C \frac{1}{z(z-1)(z+1)}\,dz = -2\pi i + \pi i + \pi i = 0$$

(3) Write the partial fraction decomposition for $\frac{z^2+2}{(z-1)(z-i)^2}$. Use this to verify directly (without using the result of section (6.2) of Lecture 6) that

$$\frac{1}{2\pi i}\int_C \frac{z^2+2}{(z-1)(z-i)^2}\,dz = 1$$

where C is a simple closed curve whose interior contains 1 and i.

Solution. Let us write $\frac{z^2+2}{(z-1)(z-i)^2} = \frac{A}{z-1} + \frac{B}{z-i} + \frac{C}{(z-i)^2}$. Then we get (by clearing denominator)

$$z^2+2 = A(z-i)^2 + B(z-1)(z-i) + C(z-1) = (A+B)z^2 + (C-2iA-(1+i)B)z + (-A+iB-C)$$

Comparing coefficients of z^0, z^1, z^2 on both sides gives:

$$A+B=1 \text{ and } -2iA-(1+i)B+C=0 \text{ and } -A+iB-C=2$$

This system can be easily solved to get:

$$A = \frac{3}{2}i \qquad B = \frac{2-3i}{2} \qquad C = \frac{-1-i}{2}$$

Now by Cauchy's integral formula, we get

$$\frac{1}{2\pi i}\int_C \frac{z^2+2}{(z-1)(z-i)^2}\,dz = A+B = 1$$

(4) Recall that in Homework 2, problem 4, we proved that

$$\left|\int_\gamma \frac{1}{z^4+9}\,dz\right| \le \frac{4\pi}{R^3}$$

where γ is the counterclockwise circle of radius R and $R^4 > 18$. Use this to prove directly that $\int_\gamma \frac{1}{z^4+9}\,dz = 0$.

Solution. By Cauchy's Theorem, the integral $\int_\gamma \frac{1}{z^4+9}\,dz$ is independent of the radius R, as long as it is sufficiently large so that γ contains all the zeroes of z^4+9. But by the inequality proved above:

$$\left|\int_\gamma \frac{1}{z^4+9}\,dz\right| \le \frac{4\pi}{R^3}$$

we see that $\left|\int_\gamma \frac{1}{z^4+9}\,dz\right|$ is smaller than any positive real number, hence it has to be zero.

(5) Let $z_1, \cdots, z_n$ be n distinct non–zero complex numbers. Let q be another non–zero complex number. Let C be a simple closed curve such that $z_1, z_2, \cdots, z_n$ and 0 are in the interior of C.

(a) Consider the function

$$f(z) = \frac{1}{z}\frac{qz-q^{-1}z_1}{z-z_1}\frac{qz-q^{-1}z_2}{z-z_2}\cdots\frac{qz-q^{-1}z_n}{z-z_n}$$

Use the result proved in class (from section (6.2) of Lecture 6), to verify that

$$\frac{1}{2\pi i}\int_C f(z)\,dz = q^n$$

Solution. This is clear, since the numerator is a polynomial of degree n, with leading coefficient q^n and the denominator is a polynomial of degree $n+1$ with leading coefficient 1. Hence the integral is the ratio $q^n/1 = q^n$ by the result proved in section (6.2) of Lecture 6.

Let $C_0, C_1, \cdots, C_n$ be small closed curves which enclose (only) $0, z_1, z_2, \cdots, z_n$ respectively. Meaning that 0 is in the interior of C_0 and none of the $z_1, \cdots, z_n$ are in the interior of C_0. Similarly z_1 is in the interior of C_1 and none of the $0, z_2, z_3, \cdots, z_n$ are in the interior of C_1, and so on. Cauchy's theorem implies that (you don't have to prove this, but it is always a good idea to convince yourself that it is true)

$$\int_C f(z)\,dz = \int_{C_0} f(z)\,dz + \int_{C_1} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

(b) Prove that

$$\frac{1}{2\pi i}\int_{C_0} f(z)\,dz = q^{-n}$$

Solution. This is again an easy application of Cauchy's integral formula:

$$\frac{1}{2\pi i}\int_{C_0} f(z)\,dz = \left[\prod_{k=1}^{n} \frac{qz - q^{-1}z_k}{z - z_k}\right]_{z=0} = q^{-n}$$

(c) Prove that for each $k = 1, 2, \cdots, n$:

$$\frac{1}{2\pi i}\int_{C_k} f(z)\,dz = (q - q^{-1}) \prod_{\substack{l=1,2,\cdots,n \\ l\neq k}} \frac{qz_k - q^{-1}z_l}{z_k - z_l}$$

Solution. This is again an easy application of Cauchy's integral formula:

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_k} f(z)\,dz &= \frac{1}{z_k}(qz_k - q^{-1}z_k) \prod_{\substack{l=1,2,\cdots,n \\ l\neq k}} \frac{qz_k - q^{-1}z_l}{z_k - z_l} \\ &= (q - q^{-1}) \prod_{\substack{l=1,2,\cdots,n \\ l\neq k}} \frac{qz_k - q^{-1}z_l}{z_k - z_l}\end{aligned}$$

(d) Put all the computations above together to see that

$$\sum_{k=1}^{n} \prod_{\substack{l=1,2,\cdots,n \\ l\neq k}} \frac{qz_k - q^{-1}z_l}{z_k - z_l} = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Solution. We have

$$\begin{aligned}q^n &= \frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\left(\int_{C_0} f(z)\,dz + \int_{C_1} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz\right) \\ &= q^{-n} + (q - q^{-1})\sum_{k=1}^{n} \prod_{\substack{l=1,2,\cdots,n \\ l\neq k}} \frac{qz_k - q^{-1}z_l}{z_k - z_l}\end{aligned}$$

Hence, we get

$$\sum_{k=1}^{n} \prod_{\substack{l=1,2,\cdots,n \\ l\neq k}} \frac{qz_k - q^{-1}z_l}{z_k - z_l} = \frac{q^n - q^{-n}}{q - q^{-1}}$$