COMPLEX VARIABLES: HOMEWORK 3

All simple closed curves considered below are assumed to be counterclockwise oriented. You can use the results proved in class, except when it is explicitly prohibited (see problems 3 and 4).

(1) Prove that $\int_C z^{-n}\, dz$ equals 0 if $n > 1$ and $2\pi i$ if $n = 1$. Here n is a natural number and C is the counterclockwise circle of radius $R > 0$.

(2) Compute all possible values of $\int_C \frac{1}{z(z^2-1)}\, dz$ for different choices of simple closed curves C which do not pass through $0, 1, -1$.

(3) Write the partial fraction decomposition for $\frac{z^2+2}{(z-1)(z-i)^2}$. Use this to verify directly (without using the result of section (6.2) of Lecture 6) that

$$\frac{1}{2\pi i}\int_C \frac{z^2+2}{(z-1)(z-i)^2}\, dz = 1$$

where C is a simple closed curve whose interior contains 1 and i.

(4) Recall that in Homework 2, problem 4, we proved that

$$\left|\int_\gamma \frac{1}{z^4+9}\, dz\right| \leq \frac{4\pi}{R^3}$$

where γ is the counterclockwise circle of radius R and $R^4 > 18$. Use this to prove directly that $\int_\gamma \frac{1}{z^4+9}\, dz = 0$.

(5) Let $z_1, \cdots, z_n$ be n distinct non–zero complex numbers. Let q be another non–zero complex number. Let C be a simple closed curve such that $z_1, z_2, \cdots, z_n$ and 0 are in the interior of C.

(a) Consider the function

$$f(z) = \frac{1}{z}\,\frac{qz - q^{-1}z_1}{z - z_1}\,\frac{qz - q^{-1}z_2}{z - z_2}\cdots\frac{qz - q^{-1}z_n}{z - z_n}$$

Use the result proved in class (from section (6.2) of Lecture 6), to verify that

$$\frac{1}{2\pi i}\int_C f(z)\, dz = q^n$$

Let $C_0, C_1, \cdots, C_n$ be small closed curves which enclose (only) $0, z_1, z_2, \cdots, z_n$ respectively. Meaning that 0 is in the interior of C_0 and none of the $z_1, \cdots, z_n$ are in the interior of C_0. Similarly z_1 is in the interior of C_1 and none of the $0, z_2, z_3, \cdots, z_n$ are in the interior of C_1, and so on. Cauchy's theorem implies that (you don't have to prove this, but it is always a good idea to convince yourself that it is true)

$$\int_C f(z)\, dz = \int_{C_0} f(z)\, dz + \int_{C_1} f(z)\, dz + \cdots + \int_{C_n} f(z)\, dz$$

(b) Prove that
$$\frac{1}{2\pi i}\int_{C_0} f(z)\,dz = q^{-n}$$

(c) Prove that for each $k = 1, 2, \cdots, n$:
$$\frac{1}{2\pi i}\int_{C_k} f(z)\,dz = (q - q^{-1}) \prod_{\substack{l=1,2,\cdots,n \\ l\neq k}} \frac{qz_k - q^{-1}z_l}{z_k - z_l}$$

(d) Put all the computations above together to see that
$$\sum_{k=1}^{n} \prod_{\substack{l=1,2,\cdots,n \\ l\neq k}} \frac{qz_k - q^{-1}z_l}{z_k - z_l} = \frac{q^n - q^{-n}}{q - q^{-1}}$$