COMPLEX VARIABLES: HOMEWORK 4

(1) Prove that the following series converge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution. We can compute the definite integral $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = 1$. The given series is bounded above by this integral, since the integral computes the area under the curve $y = 1/x^2$ from $x = 1$ to ∞, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

(b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

Solution. Since $\displaystyle\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, for $|z| < 1$, taking derivative of this gives

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1}$$

Hence we get $\displaystyle\sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}$ for $|z| < 1$. Setting $z = 1/2$ gives

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2$$

therefore, convergent.

(2) Find the radius of convergence of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{n}{2^n} z^n$

Solution. Either look at the solution of the previous problem part (b), to conclude that $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{z}{2}\right)^n$ converves for $|z/2| < 1$, and diverges for $|z/2| > 1$, hence the radius of convergence is 2. Or, we can do it directly by taking the limit of the ratio of successive terms:

$$\lim_{n\to\infty} \left| \frac{n+1}{n} \frac{2^n}{2^{n+1}} \frac{z^{n+1}}{z^n} \right| = \frac{|z|}{2}$$

Thus the radius of convergence is 2.

(b) $\sum_{n=1}^{\infty} \frac{1}{n} z^n$

Solution. By ratio test, we have

$$\lim_{n\to\infty} \left| \frac{n}{n+1} \frac{z^{n+1}}{z^n} \right| = |z|$$

Hence the radius of convergence is 1.

(3) Let $\sum_{n=1}^{\infty} c_n z^n$ be a power series with non–zero radius of convergence. Prove that the power series $\sum_{n=1}^{\infty} c_n \frac{z^{n-1}}{(n-1)!}$ has infinite radius of convergence.

Solution. There are several ways to do this. Assume R is the radius of convergence of $\sum_{n=1}^{\infty} c_n z^n$. If we assume that all c_n's are non–zero, then by ratio test we get that, for $0 < r < R$:

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| r = m < 1$$

Then applying ratio test to $\sum_{n=1}^{\infty} c_n \frac{z^{n-1}}{(n-1)!}$ gives

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n-1}} \right| \frac{|z|}{n} == \frac{m}{r} \lim_{n\to\infty} \frac{|z|}{n} = 0$$

for any value of $|z|$. Hence the radius of convergence of $\sum_{n=1}^{\infty} c_n \frac{z^{n-1}}{(n-1)!}$ is ∞.

(In order to remove the assumption that all c_n's are non–zero, we need to introduce a bit more notation. Namely, assume $1 \le i_1 < i_2 < \cdots$ are the indices where $c_{i_1}, c_{i_2}, \cdots$ are the only non–zero terms. Then the given series is $\sum_{n=1}^{\infty} c_{i_n} z^{i_n}$. The same argument as before will prove that the radius of convergence of $\sum_{n=1}^{\infty} c_{i_n} \frac{z^{i_n - 1}}{(i_n - 1)!}$ is infinity.)

Alternately, (see the proof of Abel's Theorem (8.4) of Lecture 8), for every $0 < r < R$ we know that the numbers $|c_n| r^n$ are bounded, by say $M > 0$. That is, for every $n = 1, 2, 3, \cdots$, we have $|c_n| r^n \le M$. Then the series $\sum_{n=1}^{\infty} c_n \frac{z^{n-1}}{(n-1)!}$ is dominated by the exponential series, since $|c_n| \le \frac{M}{r^n}$:

$$\sum_{n=1}^{\infty} |c_n| \frac{|z|^{n-1}}{(n-1)!} \le \frac{M}{r} \sum_{n=1}^{\infty} \frac{|z|^{n-1}}{r^{n-1}(n-1)!} = \frac{M}{r} e^{\frac{|z|}{r}}$$

which converges for all values of $|z|$. Hence we get that $\sum_{n=1}^{\infty} c_n \frac{z^{n-1}}{(n-1)!}$ has infinite radius of convergence.

(4) Find the Taylor series expansion of $\frac{z}{z^2 - 2z - 3}$ near $z = 0$. What is its radius of convergence?

Solution. Write the partial fraction decomposition using $z^2 - 2z - 3 = (z-3)(z+1)$:

$$\frac{z}{z^2 - 2z - 3} = \frac{1}{4} \left(\frac{3}{z-3} + \frac{1}{z+1} \right)$$

Now we have

$$\frac{3}{z-3} = -\frac{1}{1-(z/3)} = -\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty}(-1)^n z^n$$

Hence we get

$$\frac{z}{z^2-2z-3} = \frac{1}{4}\sum_{n=0}^{\infty}\left((-1)^n - \frac{1}{3^n}\right)z^n$$

To compute the radius of convergence, we take ratio of successive terms

$$\left|\frac{(-1)^{n+1}-3^{-n-1}}{(-1)^n-3^{-n}}z\right| \to |z| \text{ as } n\to\infty$$

Hence the radius of convergence is 1.

(5) Prove the following equation holds for $|z|<1$, and any $l = 0, 1, 2, \cdots$:

$$\frac{1}{(1-z)^{l+1}} = \sum_{n=0}^{\infty}\frac{(n+l)!}{n!l!}z^n$$

Solution. We have already checked this equation for $l = 0$. Namely, this was shown in class that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, for $|z| < 1$. Now $\frac{1}{(1-z)^{l+1}}$ is obtained from $\frac{1}{1-z}$ by taking derivative l times, and dividing by $l!$:

$$\frac{1}{(1-z)^{l+1}} = \frac{1}{l!}\left(\frac{d}{dz}\right)^l \frac{1}{1-z}$$

By Theorem (7.3) proved in the class, power series can be termwise differentiated, keeping the same radius of convergence. Hence we get (for $|z|<1$):

$$\begin{aligned}\frac{1}{(1-z)^{l+1}} &= \frac{1}{l!}\left(\frac{d}{dz}\right)^l \frac{1}{1-z} = \frac{1}{l!}\left(\frac{d}{dz}\right)^l \sum_{n=0}^{\infty} z^n \\ &= \sum_{n=l}^{\infty}\frac{n(n-1)\cdots(n-l+1)}{l!}z^{n-l} \\ &= \sum_{n=l}^{\infty}\frac{n!}{(n-l)!l!}z^{n-l} = \sum_{m=0}^{\infty}\frac{(m+l)!}{m!l!}z^m\end{aligned}$$

where in the last line, we set $m = n - l$.

(6) Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a power series with radius of convergence $R > 0$.

(a) Prove that for every r, with $0 < r < R$ we have

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty}|c_n|^2 r^{2n}$$

Solution. Within the radius of convergence, the power series can be multiplied as polynomials. So, we get:

$$|f(z)|^2 = f(z).\overline{f(z)} = \left(\sum_{n=0}^{\infty} c_n z^n\right)\left(\sum_{m=0}^{\infty} \overline{c_m}\,(\overline{z})^m\right) = \sum_{n,m} c_n\overline{c_m} z^n\,(\overline{z})^m$$

Uniform convergence implies that integral of $|f(z)|^2$ is same as sum of integrals of $c_n\overline{c_m}z^n(\overline{z})^m$. Let us compute this.

$$\frac{1}{2\pi}\int_0^{2\pi} c_n\overline{c_m} r^{n+m} e^{i\theta(n-m)}\,d\theta = \begin{cases} c_n\overline{c_n}r^{2n} & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$

(this is clear since we can easily check that for a non–zero integer l the integral $\displaystyle\int_0^{2\pi} e^{il\theta}\,d\theta = 0$).

Hence the integral to compute is

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2\,d\theta = \sum_{n=0}^{\infty}|c_n|^2 r^{2n}$$

as required.

(b) Let $M(r)$ be the absolute maximum of the function $|f(re^{i\theta})|$ for $0 \leq \theta \leq 2\pi$. Recall that we have the following inequality (Lecture 8, section (8.7)):

$$|c_n| \leq \frac{M(r)}{r^n} \text{ for every } n = 0, 1, 2, \cdots \text{ and } 0 < r < R$$

Use the previous part to prove that if there is n and r such that $|c_n| = \dfrac{M(r)}{r^n}$, then $f(z) = c_n z^n$.

Solution. Since $M(r)$ is the largest value $|f(z)|$ can take for $z = re^{i\theta}$, we get

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2\,d\theta\right| \leq \frac{1}{2\pi}M(r)^2 2\pi = M(r)^2$$

Therefore, we obtain the following inequality:

$$\sum_{n=0}^{\infty}|c_n|^2 r^{2n} \leq M(r)^2$$

Now if for some n and r, $|c_n| = M(r)/r^n$, then the corresponding term in the left–hand side of the inequality above will be

$$|c_n|^2 r^{2n} = \frac{M(r)^2}{r^{2n}} r^{2n} = M(r)^2$$

Since the other terms of the series are non–negative, in order to preserve the inequality, all other terms must be 0. Hence $c_m = 0$ for $m \neq n$ and the function $f(z)$ is just $c_n z^n$.