COMPLEX VARIABLES: HOMEWORK 7

(1) Assume $f(x)$ is a continuous function of a real variable x (defined for every $x \in \mathbb{R}$). Assume further that

- $\displaystyle\int_0^\infty f(x)\,dx$ exists (and is finite). **Meaning:** for every $\varepsilon > 0$ there exists $T > 0$ such that $\left|\displaystyle\int_T^Q f(x)\,dx\right| < \varepsilon$ for every $Q \geq T$.
- $C = \displaystyle\lim_{R\to\infty} \int_{-R}^R f(x)\,dx$ exists (and is finite). **Meaning:** for every $\varepsilon > 0$ there exists $T > 0$ such that $\left|\displaystyle\int_{-Q}^Q f(x)\,dx - C\right| < \varepsilon$ for every $Q \geq T$.

Prove that $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ exists and is equal to C.

Solution. We need to prove that given any $\varepsilon > 0$, we can find $T > 0$ such that $\left|\displaystyle\int_{-S}^R f(x)\,dx - C\right| < \varepsilon$ for every $R, S \geq T$. By what is given, we can always find $T_1, T_2 > 0$ so that

$$\left|\int_{R_1}^{R_2} f(x)\,dx\right| < \frac{\varepsilon}{2} \qquad \left|\int_{-S}^{S} f(x)\,dx - C\right| < \frac{\varepsilon}{2}$$

as long as $R_1, R_2 \geq T_1$ and $S \geq T_2$. Now take $T = \text{maximum}(T_1, T_2)$. Then for every $R, S \geq T$ we will have

$$\left|\int_{-S}^{R} f(x)\,dx - C\right| \leq \left|\int_{-S}^{S} f(x)\,dx - C\right| + \left|\int_{S}^{R} f(x)\,dx\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as required.

(2) In the following steps, prove that $\displaystyle\int_0^\infty \frac{x\cos(x)}{x^2 - 2x + 10}\,dx$ exists.

(a) Let $b_1, b_2, \cdots$ be real numbers, such that $b_1 \geq b_2 \cdots \geq 0$. Assume that $\displaystyle\lim_{n\to\infty} b_n = 0$. Then prove that $\displaystyle\sum_{n=1}^\infty (-1)^{n-1} b_n$ converges.

Solution. This is basically the alternating series test, which we prove as follows. Note that we have

$$b_n - b_{n+1} + b_{n+2} - \cdots + (-1)^m b_{n+m} \leq b_n$$

This is because for m even we can group the terms on the left hand side as

$$b_n - (b_{n+1} - b_{n+2}) - \cdots - (b_{n+m-1} - b_{n+m}) \leq b_n$$

(since all terms in the parantheses are non-negative). Similarly when m is odd we can group the terms as

$$b_n - (b_{n+1} - b_{n+2}) - \cdots - (b_{n+m-2} - b_{n+m-1}) - b_m \leq b_n$$

(again all terms being subtracted from b_n are non-negative, including b_m). Since we are given that $b_n \to 0$ as $n \to \infty$, this proves that given any $\varepsilon > 0$ we can find n such that $b_n < \varepsilon$. Then for every $m \geq 1$ we will have

$$|b_n - b_{n+1} + b_{n+2} - \cdots + (-1)^m b_{n+m}| \leq b_n < \varepsilon$$

Hence the alternating series converges.

(b) Prove that $\dfrac{x}{x^2 - 2x + 10}$ is a decreasing function of x for $|x| > \sqrt{10}$.

Solution. Take the derivative of this function $f(x) = \dfrac{x}{x^2 - 2x + 10}$:

$$f'(x) = \frac{-x^2 + 10}{(x^2 - 2x + 10)^2} < 0 \text{ for } x^2 > 10$$

Thus the function $f(x)$ is positive and decreasing for $x > \sqrt{10}$.

(c) Recall that $\cos(x)$ for $x \in \left[\dfrac{(2n+1)\pi}{2}, \dfrac{(2n+3)\pi}{2}\right]$ is positive if n is odd and negative if n is even. Define real numbers c_n by

$$(-1)^{n-1} c_n = \int_{(2n+1)\pi/2}^{(2n+3)\pi/2} \frac{x\cos(x)}{x^2 - 2x + 10}\, dx$$

Prove that $c_1 \geq c_2 \geq \cdots \geq 0$ and that $\lim\limits_{n\to\infty} c_n = 0$.

Solution. It is quite easy to see that $c_n \to 0$ as $n \to \infty$. To save space, let me write I_n for the closed interval $\left[\frac{(2n+1)\pi}{2}, \frac{(2n+3)\pi}{2}\right]$. Then for $x \in I_n$, we have $0 < f(x) \leq f((2n+1)\pi/3)$. Also, $|\cos(x)| \leq 1$. Therefore, we get

$$c_n = \left|\int_{I_n} f(x)\cos(x)\, dx\right| \leq f((2n+1)\pi/3).\text{length}(I_n) = f((2n+1)\pi/3).\pi$$

But $f((2n+1)\pi/3) \to 0$ as $n \to \infty$. Hence same is true for $\{c_n\}$.

Now to prove $c_{n+1} \leq c_n$ we can make use of substitution $y = x - \pi$:

$$\begin{aligned} c_{n+1} &= (-1)^n \int_{I_{n+1}} f(x)\cos(x)\, dx = (-1)^n \int_{I_n} f(y+\pi)\cos(y+\pi)\, dy \\ &= -(-1)^n \int_{I_n} f(y+\pi)\cos(y)\, dy \leq (-1)^{n-1} \int_{I_n} f(y)\cos(y)\, dy = c_n \end{aligned}$$

where we used the fact that f is decreasing, so $f(y+\pi) \leq f(y)$.

(d) Use part (a) to conclude that $\sum_{n=1}^{\infty} (-1)^{n-1} c_n$ exists and that it equals $\displaystyle\int_{3\pi/2}^{\infty} \frac{x\cos(x)}{x^2 - 2x + 10}\, dx$.

Solution. This is obvious because

$$\int_{3\pi/2}^{\infty} \frac{x\cos(x)}{x^2 - 2x + 10}\, dx = \sum_{n=1}^{\infty} \int_{(2n+1)\pi/2}^{(2n+3)\pi/2} \frac{x\cos(x)}{x^2 - 2x + 10}\, dx = \sum_{n=1}^{\infty} (-1)^{n-1} c_n$$

and the latter converges by the alternating series test from part (a).

(3) Prove that the Laplace transform of $\dfrac{t^n}{n!}$ (where $n \geq 0$ is an integer) is z^{-n-1}, for $\text{Re}(z) > 0$. (recall that the Laplace transform of a function $\varphi(t)$ of a real variable t is given by

$\displaystyle\int_0^\infty \varphi(t)e^{-zt}\,dt.$)

Solution. For $n = 0$ this statement was proved in class:

$$\int_o^\infty e^{-tz}\,dt = \left[\frac{e^{-zt}}{-z}\right]_{t=0}^{t=\infty} = \frac{1}{z} \text{ if } \operatorname{Re}(z) > 0$$

because for $\operatorname{Re}(z) > 0$, we have $\lim_{t\to\infty} e^{-zt} = 0$ (which would be false if $\operatorname{Re}(z)$ were non-positive).

Now for $n \geq 1$ we have (using integration by parts):

$$\int \frac{t^n}{n!}e^{-zt}\,dt = \frac{t^n}{n!}\frac{e^{-zt}}{-z} + \frac{1}{z}\int \frac{t^{n-1}}{(n-1)!}e^{-zt}\,dt$$

taking the limits $t = 0$ and $t = \infty$, and again using that for every $n \geq 1$, $\lim_{t\to\infty} t^n e^{-zt} = 0$ as long as $\operatorname{Re}(z) > 0$, we get

$$\int_0^\infty \frac{t^n}{n!}e^{-zt}\,dt = \frac{1}{z}\int_0^\infty \frac{t^{n-1}}{(n-1)!}e^{-zt}\,dt$$

and we are done by induction.