# COMPLEX VARIABLES: HOMEWORK 7

(1) Assume $f(x)$ is a continuous function of a real variable $x$ (defined for every $x \in \mathbb{R}$). Assume further that

- $\displaystyle\int_0^\infty f(x)\,dx$ exists (and is finite). **Meaning:** for every $\varepsilon > 0$ there exists $T > 0$ such that $\displaystyle\left|\int_T^Q f(x)\,dx\right| < \varepsilon$ for every $Q \geq T$.
- $\displaystyle C = \lim_{R\to\infty} \int_{-R}^R f(x)\,dx$ exists (and is finite). **Meaning:** for every $\varepsilon > 0$ there exists $T > 0$ such that $\displaystyle\left|\int_{-Q}^Q f(x)\,dx - C\right| < \varepsilon$ for every $Q \geq T$.

Prove that $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ exists and is equal to $C$.

(2) In the following steps, prove that $\displaystyle\int_0^\infty \frac{x\cos(x)}{x^2-2x+10}\,dx$ exists.

(a) Let $b_1, b_2, \cdots$ be real numbers, such that $b_1 \geq b_2 \cdots \geq 0$. Assume that $\lim_{n\to\infty} b_n = 0$. Then prove that $\displaystyle\sum_{n=1}^\infty (-1)^{n-1} b_n$ converges.

(b) Prove that $\dfrac{x}{x^2-2x+10}$ is a decreasing function of $x$ for $|x| > \sqrt{10}$.

(c) Recall that $\cos(x)$ for $x \in \left[\dfrac{(2n+1)\pi}{2}, \dfrac{(2n+3)\pi}{2}\right]$ is positive if $n$ is odd and negative if $n$ is even. Define real numbers $c_n$ by

$$(-1)^{n-1} c_n = \int_{(2n+1)\pi/2}^{(2n+3)\pi/2} \frac{x\cos(x)}{x^2-2x+10}\,dx$$

Prove that $c_1 \geq c_2 \geq \cdots \geq 0$ and that $\lim_{n\to\infty} c_n = 0$.

(d) Use part (a) to conclude that $\sum_{n=1}^\infty (-1)^{n-1} c_n$ exists and that it equals $\displaystyle\int_{3\pi/2}^\infty \frac{x\cos(x)}{x^2-2x+10}\,dx$.

(3) Prove that the Laplace transform of $\dfrac{t^n}{n!}$ (where $n \geq 0$ is an integer) is $z^{-n-1}$, for $\mathrm{Re}(z) > 0$. (recall that the Laplace transform of a function $\varphi(t)$ of a real variable $t$ is given by $\displaystyle\int_0^\infty \varphi(t)e^{-zt}\,dt$.)