COMPLEX VARIABLES: HOMEWORK 8

Recall that we defined the following two functions:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt \quad Re(z) > 0$$

$$\Gamma_1(z) = \frac{1}{ze^{\gamma z}\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right)}$$

We have not yet proved that these two are the same. In the following problems, we are only concerned with $\Gamma_1(z)$.

(1) In Lecture 16, page 5, the value of $\Gamma_1'(1)$ is computed (it is $-\gamma$). What is the value of $\Gamma_1'(n)$ where $n = 2, 3, 4, \cdots$.

(2) Find the residue $\text{Res}_{z=-n}\left(\Gamma_1(z)\right)$ where $n = 0, 1, 2, 3, \cdots$.

(3) Prove that $\Gamma_1(z) = \lim_{n\to\infty} \frac{(n-1)!}{z(z+1)\cdots(z+n-1)}.n^z$ where $n^z = e^{z\ln(n)}$.

(4) Let $\Psi(z) = \frac{\Gamma_1'(z)}{\Gamma_1(z)}$. Prove that

$$\Psi(z+1) = \Psi(z) + \frac{1}{z}$$

Use this to solve the following difference equation:

$$f(z+1) = f(z) + \frac{1}{z^2 - 3z + 2}$$

(5) Let $\Psi(z)$ be as in the previous problem. Prove that

$$\Psi(z) - \Psi(1-z) = -\pi\cot(\pi z)$$

(6) Let $A(z) = \prod_{n=1}^{N}\left(\frac{z-a_n}{z-b_n}\right)$, where $a_1, b_1, a_2, b_2, \cdots, a_N, b_N$ are complex numbers. Find a solution of $F(z+1) = A(z)F(z)$. Where are its zeroes and poles?