COMPLEX VARIABLES: HOMEWORK 9

The problems below concern the gamma and the psi function, defined as:

$$\Gamma(z) = \frac{1}{ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}} \qquad \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

(1) Use Theorem 16.8 of Lecture 16 (page 9) to prove that

$$e^z - 1 = ze^{\frac{z}{2}} \prod_{n=\pm1,\pm2,\cdots} \left(1 - \frac{z}{2\pi n i}\right) e^{\frac{z}{2\pi n i}}$$

Solution. Consider the function $F(z) = \dfrac{e^z-1}{z}$. This function is holomorphic on the entire complex plane and has zeroes (each of multiplicity 1) at $z = 2\pi i k$ where $k = \pm1, \pm2, \cdots$. Clearly $F(0) = 1$ and we can compute its logarithmic derivative as:

$$f(z) = \frac{F'(z)}{F(z)} = \frac{e^z}{e^z-1} - \frac{1}{z} = \frac{ze^z - e^z + 1}{z(e^z-1)}$$

Thus we get $f(0) = \lim_{z\to 0} \dfrac{ze^z - e^z + 1}{z(e^z-1)} = \dfrac{1}{2}$. Applying Theorem 16.8 we have:

$$F(z) = F(0)e^{\frac{F'(0)}{F(0)}z} \prod_{n=\pm1,\pm2,\cdots} \left(1 - \frac{z}{2\pi n i}\right) e^{\frac{z}{2\pi n i}}$$

$$e^z - 1 = ze^{\frac{z}{2}} \prod_{n=\pm1,\pm2,\cdots} \left(1 - \frac{z}{2\pi n i}\right) e^{\frac{z}{2\pi n i}}$$

(2) Use Lecture 16 page 5, to prove that

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$$

Solution. According to the formula given in Lecture 16, we have

$$\Psi(z) = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty} \left(\frac{-1}{z+n} + \frac{1}{n}\right)$$

Therefore, differentiating termwise again we get:

$$\Psi'(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{(z+n)^2}$$

(3) Use Gauß' formula:

$$\Psi(z) = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt$$

to prove that

$$\Psi(1) - \Psi\left(\frac{1}{2}\right) = 2\ln(2)$$

Solution. Using the given formula we have:

$$\begin{aligned}\Psi(1)-\Psi\left(\frac{1}{2}\right) &= \int_0^\infty \left(\frac{e^{-t}}{t}-\frac{e^{-t}}{1-e^{-t}}\right)dt \\ &\quad -\int_0^\infty \left(\frac{e^{-t}}{t}-\frac{e^{-\frac{t}{2}}}{1-e^{-t}}\right)dt \\ &= \int_0^\infty \frac{e^{-t/2}-e^{-t}}{1-e^{-t}}\,dt \\ &= \int_0^\infty e^{-t/2}\frac{1-e^{-t/2}}{(1-e^{-t/2})(1+e^{-t/2})}\,dt \\ &= \int_0^\infty \frac{e^{-t/2}}{1+e^{-t/2}}\,dt = \left[-2\ln(1+e^{-t/2})\right]_0^\infty \\ &= -2\ln(1)-(-2\ln(2)) = 2\ln(2)\end{aligned}$$

(4) Recall that we defined $B(p,q)=\displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$. Prove that (for $\mathrm{Re}(z)>0$):

$$\Gamma(z)=\lim_{n\to\infty} B(z,n)n^z$$

(Hint: problem 3 of homework 8).

Solution. Since we prove that $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ we get:

$$n^zB(z,n)=n^z\frac{\Gamma(z)\Gamma(n)}{\Gamma(z+n)}=n^z\frac{\Gamma(z).(n-1)!}{(z+n-1)\cdots(z+1)z\Gamma(z)}$$

Therefore the limit in question is:

$$\lim_{n\to\infty} n^z\frac{(n-1)!}{z(z+1)\cdots(z+n-1)}=\Gamma(z)$$

by Problem 3 of homework 8.

(5) Recall that we defined the numbers $b_0,b_1,b_2,\cdots$ as the coefficients of the Taylor series

$$\frac{t}{e^t-1}=\sum_{n=0}^\infty \frac{b_n}{n!}t^n$$

(It was stated in the class that $b_0=1$ and $b_1=-1/2$). Prove that these numbers satisfy the following relation, for each $n\geq 2$:

$$\sum_{k=0}^{n-1}\frac{b_k}{k!(n-k)!}=0$$

Solution. Clear the denominator in $\dfrac{t}{e^t-1}=\displaystyle\sum_{n=0}^\infty \frac{b_n}{n!}t^n$, to get

$$t=\left(e^t-1\right)\left(\sum_{k=0}^\infty \frac{b_k}{k!}t^k\right)=\left(\sum_{l=1}^\infty \frac{t^l}{l!}\right)\left(\sum_{k=0}^\infty \frac{b_k}{k!}t^k\right)$$

Now the coefficient of t^n in the right–hand side is: $\displaystyle\sum_{k=1}^{n-1}\frac{b_k}{k!}.\frac{1}{(n-k)!}$. Therefore, for $n\geq 2$ it must be zero.