COMPLEX VARIABLES: HOMEWORK 9

The problems below concern the gamma and the psi function, defined as:

$$\Gamma(z) = \frac{1}{ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}} \qquad\qquad \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

(1) Use Theorem 16.8 of Lecture 16 (page 9) to prove that

$$e^z - 1 = ze^{\frac{z}{2}} \prod_{n=\pm 1,\pm 2,\cdots} \left(1 - \frac{z}{2\pi n i}\right) e^{\frac{z}{2\pi n i}}$$

(2) Use Lecture 16 page 5, to prove that

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$$

(3) Use Gauß' formula:

$$\Psi(z) = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt$$

to prove that

$$\Psi(1) - \Psi\left(\frac{1}{2}\right) = 2\ln(2)$$

(4) Recall that we defined $B(p,q) = \displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$. Prove that (for $\mathrm{Re}(z) > 0$):

$$\Gamma(z) = \lim_{n\to\infty} B(z,n)n^z$$

(Hint: problem 3 of homework 8).

(5) Recall that we defined the numbers $b_0, b_1, b_2, \cdots$ as the coefficients of the Taylor series

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{b_n}{n!} t^n$$

(It was stated in the class that $b_0 = 1$ and $b_1 = -1/2$). Prove that these numbers satisfy the following relation, for each $n \geq 2$:

$$\sum_{k=0}^{n-1} \frac{b_k}{k!(n-k)!} = 0$$