Lecture 0

Complex Numbers

(0.1) Definition : A complex number is given by a pair of real numbers (a, b). It is denoted by $z = a + b i$

iota

- Addition of complex numbers is componentwise : if $z_1 = a + bi$ and $z_2 = c + di$ then

$$z_1 + z_2 = (a+c) + (b+d)i$$

Note : the set of all complex numbers is same as $\mathbb{R}^2$ (2 dimensional vector space).

- Multiplication of complex numbers : if $z_1 = a + bi$ and $z_2 = c + di$ then

$$z_1 z_2 := (ac - bd) + (ad + bc)i$$

- Magnitude of a complex number : for $z = a + bi$

$$|z| = \sqrt{a^2 + b^2} \quad \text{(a non-negative real number)}$$

Note : $|z|$ = distance between $(0,0)$ and (a,b) in $\mathbb{R}^2$.

Notation : $\mathbb{C}$ = set of all complex numbers (equipped with structures/operations of addition, multiplication and magnitude defined above)

(0.2) Remark: multiplication of complex numbers is easily remembered as follows:

(i) multiplication distributes over addition

(ii) $\boxed{i^2 = -1}$

e.g. $(1+2i)(3-4i) = 1(3) + 1(-4i) + (2i)3 + (2i)(-4i)$

$= 3 - 4i + 6i - 8i^2 \leftarrow = -1$

$= 3 - 4i + 6i + 8 = 11 + 2i$

(0.3) The usual properties of addition and multiplication of real numbers hold for complex numbers as well:

for z, z_1, z_2, z_3 complex numbers,

(a) $z_1 + z_2 = z_2 + z_1$, $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$0 + z = z = z + 0$ (here $0 = 0 + 0i$)

(b) $z_1 z_2 = z_2 z_1$, $z_1(z_2 z_3) = (z_1 z_2) z_3$

$0 \cdot z = 0$ and $1 \cdot z = z$ (here $1 = 1 + 0i$)

Moreover for $z \neq 0$, we have a complex number (unique) z^{-1} such that $z z^{-1} = 1$. z^{-1} is given as follows:

Definition: Conjugate of $z = a + bi$ is defined as

$$\overline{z} = a - bi$$

Note $z\overline{z} = (a+bi)(a-bi) = a^2 - b^2 i^2$

$= a^2 + b^2 = |z|^2$

$$\boxed{z\bar{z} = |z|^2}$$

Hence, if $|z| \neq 0$, we get $z\left(\dfrac{\bar{z}}{|z|^2}\right) = 1$. So, inverse of z is given by

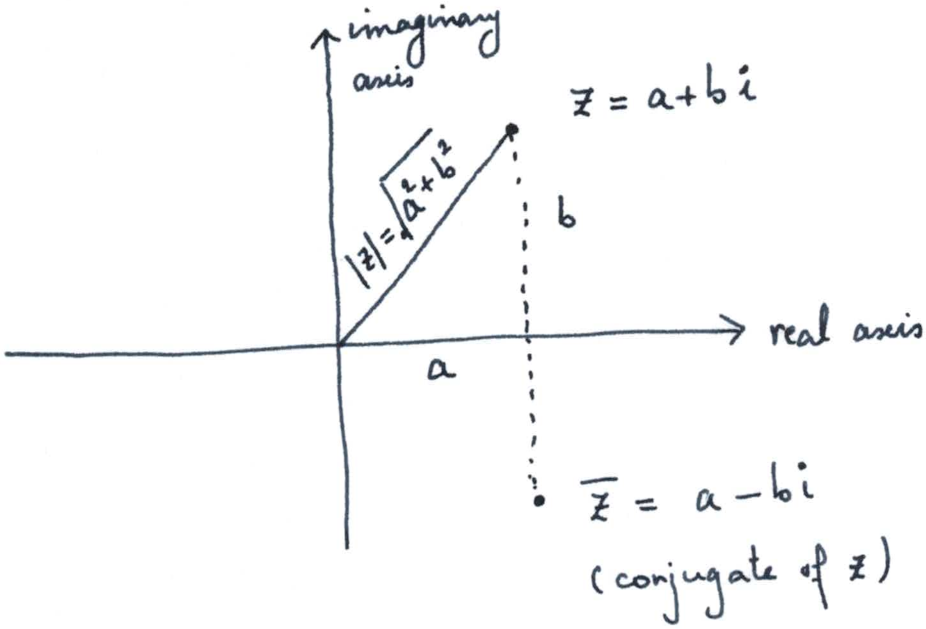
$$\boxed{z^{-1} \text{ or } \frac{1}{z} = \frac{\bar{z}}{|z|^2}}$$

(0.4) Some more definitions : for $z = a + bi$ a complex number

a is called <u>real part of z</u>, denoted by $\mathrm{Re}(z)$

b is called <u>imaginary part of z</u>, denoted by $\mathrm{Im}(z)$

Picture of $\mathbb{C}$ (complex plane)



Note :

$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$$

$$\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

e.g. find real and imaginary parts of $\dfrac{1+2i}{1-i}$

$$\frac{1+2i}{1-i} = \frac{(1+2i)(1+i)}{(1-i)(1+i)} = \frac{-1+3i}{2} = -\frac{1}{2} + \frac{3}{2}i$$

Real part $= -\frac{1}{2}$ Imaginary part $= \frac{3}{2}$

(0.5) Quadratic equations : recall that solutions of a quadratic equation $Ax^2 + Bx + C = 0$ $(A \neq 0)$ are given by

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

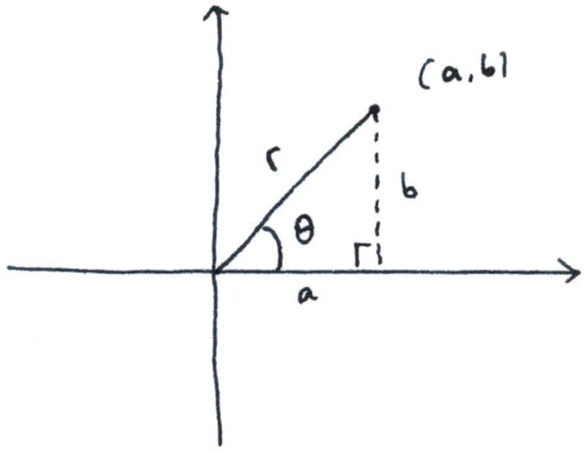
The answer always makes sense as a complex number, for any real numbers A, B and C (also for any complex numbers A, B and C as we will see in a moment)

e.g. $x^2 + x + 1 = 0$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}\, i}{2} \quad (\text{recall } i^2 = -1)$$

(0.6) Polar coordinates. Recall the change of cartesian coordinates to polar coordinates



$$r = \sqrt{a^2 + b^2}$$

$$\boxed{\begin{aligned} \cos(\theta) &= \frac{a}{\sqrt{a^2+b^2}} \\ \sin(\theta) &= \frac{b}{\sqrt{a^2+b^2}} \end{aligned}}$$

uniquely determine θ up to adding an integer multiple of 2π.

Conversely $a = r\cos(\theta)$ and $b = r\sin(\theta)$.

Thus a complex number can be equivalently written in polar form

$$\begin{aligned} z &= a + bi \\ &= r\left(\cos(\theta) + \sin(\theta)\, i\right) \end{aligned}$$

$r = |z| =$ magnitude or modulus of z

$\theta = \arg(z)$ called argument of z (be careful: it is only defined up to adding integer multiple of 2π)

(0.7) Multiplication in polar form:

if $z_1 = r_1(\cos(\theta_1) + \sin(\theta_1) i)$ and

$z_2 = r_2(\cos(\theta_2) + \sin(\theta_2) i)$, then

$$\boxed{z_1 z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2) i)}.$$ In other words:

$$|z_1 z_2| = |z_1|\,|z_2| \quad \text{and} \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Proof. Recall the addition formulae

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Now $$z_1 z_2 = r_1 r_2 \Big[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + \big(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\big) i\Big]$$

$$= r_1 r_2 \big[\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2) i\big]$$ as desired $\square$

(0.8) n^{th} roots of a complex number.

Fix a complex number $w = s(\cos(\phi) + \sin(\phi) i)$ and a positive integer n. Let us try to solve for z such that $z^n = w$.

If $z = r(\cos(\theta) + \sin(\theta) i)$ is any such complex number

then $$\underbrace{r^n(\cos(n\theta) + \sin(n\theta) i)}_{\overset{\shortparallel}{z^n}} = s(\cos(\phi) + \sin(\phi) i)$$

Hence $\boxed{r^n = s}$ and $\boxed{\begin{aligned} \cos(n\theta) &= \cos(\phi) \\ \sin(n\theta) &= \sin(\phi) \end{aligned}}$

$\downarrow$ $r = s^{1/n}$

$\downarrow$ $n\theta$ and ϕ differ by an integer multiple of 2π

Therefore we have many possibilities for θ, namely

$$n\theta = \phi \implies \theta = \frac{1}{n}\phi$$

$$n\theta = \phi + 2\pi \implies \theta = \frac{1}{n}\phi + \frac{2\pi}{n}$$

$$n\theta = \phi + 2(2\pi) \implies \theta = \frac{1}{n}\phi + 2\left(\frac{2\pi}{n}\right)$$

$$\vdots \qquad\qquad \vdots$$

$$n\theta = \phi + (n-1)(2\pi) \implies \theta = \frac{1}{n}\phi + (n-1)\left(\frac{2\pi}{n}\right)$$

$$n\theta = \phi + n(2\pi) \implies \theta = \frac{1}{n}\phi + n\left(\frac{2\pi}{n}\right) = \frac{1}{n}\phi + 2\pi$$

give the same complex number

There are exactly n possibilities for θ, namely

$$\frac{1}{n}\phi, \quad \frac{1}{n}\phi + \frac{2\pi}{n}, \quad \frac{1}{n}\phi + 2\left(\frac{2\pi}{n}\right), \ldots, \quad \frac{1}{n}\phi + (n-1)\left(\frac{2\pi}{n}\right)$$

So we conclude that there are exactly n solutions of the original equation $z^n = w$.

e.g. take $w = 1$. That is, find all complex numbers z
$n = 6$

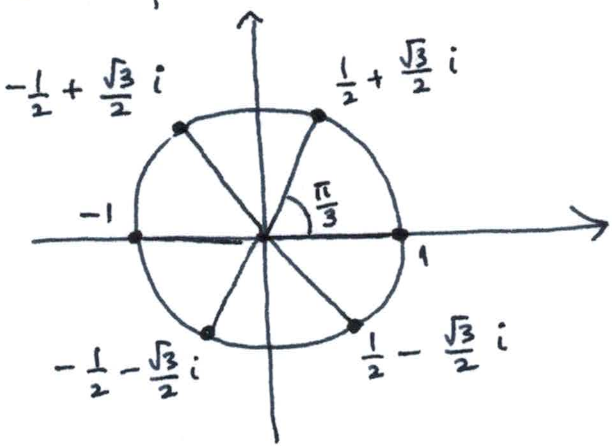
such that $z^6 = 1$.

Sol. $|z| = 1$ and $\arg(z)$ could be one of the following

$$0, \ \frac{2\pi}{6}, \ \frac{4\pi}{6}, \ \frac{6\pi}{6}, \ \frac{8\pi}{6}, \ \frac{10\pi}{6}, \ \frac{12\pi}{6}$$

$|z| = 1$ is equation of a circle of radius 1.

Picture of six roots of $z^6 - 1$:



(0.9) Fundamental Theorem of Algebra

Every polynomial equation over $\mathbb{C}$:

$$c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n = 0 \qquad (c_n \neq 0)$$

has a solution in $\mathbb{C}$.