Lecture 1

(1.0) Recall that we defined the set of complex numbers $\mathbb{C}$ as $\mathbb{R}^2$ together with the following operations

(i) Addition (componentwise) (ii) Modulus of a complex number

(iii) Multiplication by a real number, which is a special case of

(iv) multiplication of two complex numbers

Note: (i) – (iii) structures are same as those on $\mathbb{R}^2$ = 2 dimensional real vector space from Calculus III. The only new thing we introduced is the multiplication of two complex numbers.

Triangle inequality: for any two complex numbers z_1 and z_2, we have

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof: already appeared in calculus III. Recall the formula for vectors $\vec{u}$ and $\vec{v}$

$$\begin{aligned} |\vec{u}+\vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}|\cos\theta \\ &\leq |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}| \quad (\text{because } \cos\theta \leq 1) \\ &= (|\vec{u}| + |\vec{v}|)^2 \end{aligned}$$

Hence the inequality.

(1.1) Functions of a complex variable.

In this lecture we introduce functions from a domain which is a subset of the complex numbers to complex numbers.

$$f : D \longrightarrow \mathbb{C} \quad \text{where } D \text{ is a subset of } \mathbb{C}.$$

For us D will always be <u>an open set</u>.

<u>Definition</u>. A subset D of $\mathbb{C}$ is called <u>open</u> if for every point z in D, there exists a real positive number $r > 0$, such that the open disc around z, of radius r, is contained in D.

That is, if we define

$$B_r(z) = \text{set of all complex numbers } w \text{ such that } |w - z| < r$$

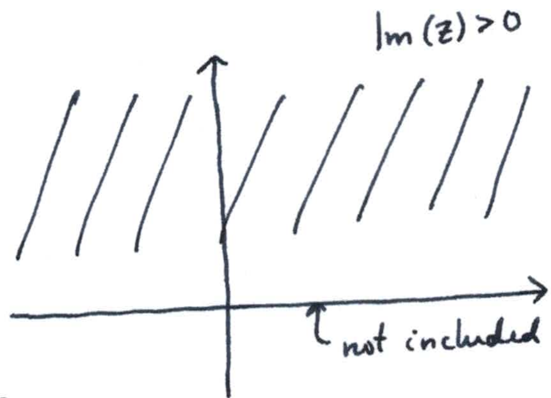
then $B_r(z) \subset D$.

e.g. (i) $D = \mathbb{C}$ is open. Also $D =$ empty set is open (but we wont worry about it)

(ii) $D =$ set of complex numbers such that the imaginary part is positive

$= \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$

is open

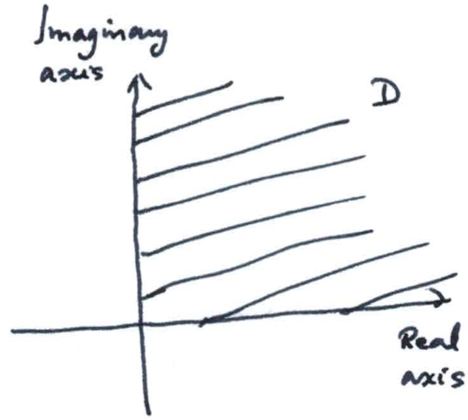


(iii) $D = \mathbb{R} \subset \mathbb{C}$ is <u>not</u> open.
(real line)

(iv) $D =$ first quadrant

$= \{ z \mid \operatorname{Re}(z) \geq 0 \text{ and } \operatorname{Im}(z) \geq 0 \}$

is not open.



(1.2) Limits & continuity.

Let D be an open subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be a function. Let z_0 be a point in D.

<u>Definition</u> $\lim_{z \to z_0} f(z) = L$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |z - z_0| < \delta$, we have

$$|f(z) - L| < \varepsilon.$$

We say f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

This definition is nothing new. Since writing real and imaginary parts of f separately

$$f(x + y i) = u(x,y) + i\, v(x,y)$$

we see that f is nothing but two real-valued functions of 2 real variables: $u(x,y)$ and $v(x,y)$ defined on the domain D.

- if $z_0 = x_0 + i y_0$ then

$$\lim_{z \to z_0} f(z) = \left(\lim_{(x,y) \to (x_0, y_0)} u(x,y) \right) + i \left(\lim_{(x,y) \to (x_0,y_0)} v(x,y) \right)$$

- f is continuous at z_0 if and only if $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0)

→ These notions were already studied in Calculus III.

However the notion of complex differentiable is going to be something new.

(1.3) Definition. $f : D \to \mathbb{C}$ as before and $z_0 \in D$.

We say f is <u>$\mathbb{C}$ - differentiable</u> at z_0 if

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \quad \text{exists} \quad (\text{denoted by } f'(z_0))$$

(caution: here h is a complex number)

Caution: Notion of $\mathbb{C}$ - differentiable is stronger that real differentiable. Meaning, if we write $f(x+iy) = u(x,y) + i v(x,y)$, that it is possible that both $u(x,y)$ and $v(x,y)$ are differentiable at a point, but f is not $\mathbb{C}$ - differentiable.

eg. let $f(z) = \mathrm{Re}(z)$. That is, $u(x,y) = x$ and $v(x,y) = 0$.

Then u and v are differentiable everywhere. But

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h \to 0} \frac{\mathrm{Re}(h)}{h} = \lim_{\substack{(h_1,h_2) \to (0,0) \\ h_1, h_2 \in \mathbb{R}}} \frac{h_1}{h_1 + i h_2}$$

does not exist.

Conclusion: f is $\mathbb{C}$ - differentiable implies u and v are differentiable but the converse is false.

(1.4) Theorem (Cauchy-Riemann).

$$\left[\begin{array}{l} f : D \to \mathbb{C} \text{ as before written as } f(x+iy) = u(x,y) + i\,v(x,y) \\ z_0 = x_0 + i y_0 \in D \end{array}\right]$$

f is $\mathbb{C}$-differentiable at z_0 if and only if the following two conditions hold

(i) $u(x,y)$ and $v(x,y)$ are differentiable at (x_0, y_0)

(ii) $$\left.\begin{array}{l} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{array}\right\} \text{ Cauchy-Riemann equations.}$$

at (x_0, y_0)

In this case $f'(z_0) = u_x(x_0, y_0) - i\, u_y(x_0, y_0)$.

Proof. Let us assume that f is differentiable at z_0; and prove that Cauchy-Riemann equations hold.

f is differentiable at $z_0 \Rightarrow \lim\limits_{h \to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$ exists $(= f'(z_0))$

if $h = h_1 + i h_2$ then

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{(h_1,h_2)\to(0,0)} \left[\frac{u(x_0+h_1, y_0+h_2) - u(x_0,y_0)}{h_1 + i h_2} + i\, \frac{v(x_0+h_1, y_0+h_2) - v(x_0,y_0)}{h_1 + i h_2} \right]$$

- Set $h_2 = 0$ and let $h_1 \to 0$:

  we get $f'(z_0) = u_x(x_0,y_0) + i\, v_x(x_0,y_0)$

must be equal

- Set $h_1 = 0$ and let $h_2 \to 0$:

  we get $f'(z_0) = \dfrac{1}{i} u_y(x_0,y_0) + v_y(x_0,y_0)$
  $= -i\, u_y(x_0,y_0) + v_y(x_0,y_0)$

Hence $u_x(x_0,y_0) + i\, v_x(x_0,y_0) = v_y(x_0,y_0) - i\, u_y(x_0,y_0)$

$\Rightarrow \quad u_x = v_y$ and $v_x = -u_y$ (C-R equations)

and $f'(z_0) = u_x(x_0,y_0) + i\, v_x(x_0,y_0)$

$= u_x(x_0,y_0) - i\, u_y(x_0,y_0)$ as claimed.

The converse is a bit longer. We are given that u_x, u_y, v_x, v_y exist and are continuous at (x_0,y_0). And that C-R equations hold.

To prove: $\displaystyle\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = u_x(x_0,y_0) - i\, u_y(x_0,y_0)$

That is, given $\varepsilon > 0$, we need to find δ such that, whenever $0 < |h| < \delta$, we have $\boxed{\left|\dfrac{f(z_0+h)-f(z_0)}{h} - \left(u_x(x_0,y_0) - i\, u_y(x_0,y_0)\right)\right| < \varepsilon}$

Let us begin by unfolding the left-hand side of this inequality by clearing the denominator

$$\text{L.H.S.} = \frac{1}{|h|} \cdot \left| f(z_0+h) - f(z_0) - h\,(u_x - i\,u_y) \right|$$

$$= \frac{1}{|h|} \cdot \left| f(z_0+h) - f(z_0) - h_1 u_x - h_2 u_y + i\,(h_1 u_y - h_2 u_x) \right|$$

(here we write $h = h_1 + i h_2$ as before)

$$= \frac{1}{|h|} \left| f(z_0+h) - f(z_0) - h_1 u_x - h_2 u_y - i\,(h_1 v_x + h_2 v_y) \right|$$

(by C-R equations)

$$= \frac{1}{|h|} \cdot \Big| u(x_0+h_1, y_0+h_2) - u(x_0,y_0) - h_1 u_x(x_0,y_0) - h_2 u_y(x_0,y_0)$$
$$+ i\big(v(x_0+h_1, y_0+h_2) - v(x_0,y_0) - h_1 v_x(x_0,y_0) - h_2 v_y(x_0,y_0)\big)\Big|$$

$$\leq \frac{1}{|h|} \cdot \Big[\big| u(x_0+h_1, y_0+h_2) - u(x_0,y_0) - h_1 u_x(x_0,y_0) - h_2 u_y(x_0,y_0) \big|$$
$$+ \big| v(x_0+h_1, y_0+h_2) - v(x_0,y_0) - h_1 v_x(x_0,y_0) - h_2 v_y(x_0,y_0) \big| \Big]$$

(by triangle inequality $|\alpha + i\beta| \leq |\alpha| + |i\beta| = |\alpha| + |\beta|$)

The remainder is standard argument. We treat u-terms and v-term separately.

$$\big| u(x_0+h_1, y_0+h_2) - u(x_0,y_0) - h_1 u_x(x_0,y_0) - h_2 u_y(x_0,y_0) \big|$$
$$= \Big| u(x_0+h_1, y_0+h_2) - u(x_0+h_1, y_0) - h_2 u_y(x_0+h_1, y_0)$$
$$+ u(x_0+h_1, y_0) - u(x_0,y_0) - h_1 u_x(x_0,y_0)$$
$$+ h_2 \big(u_y(x_0+h_1, y_0) - u_y(x_0,y_0)\Big|$$

$$\leq \big| u(x_0+h_1, y_0+h_2) - u(x_0+h_1, y_0) - h_2 u_y(x_0+h_1, y) \big|$$
$$+ \big| u(x_0+h_1, y_0) - u(x_0,y_0) - h_1 u_x(x_0,y_0) \big|$$
$$+ \big| h_2 \big(u_y(x_0+h_1, y_0) - u_y(x_0,y_0)\big) \big|$$

Now given $\varepsilon > 0$, we can find $\delta_1, \delta_2, \delta_3 > 0$ such that

$$|h_2| < \delta_1 \implies \big| u(x_0+h_1, y_0+h_2) - u(x_0+h_1, y_0) - h_2 u_y(x_0+h_1, y) \big| < \frac{\varepsilon}{6} |h_2|$$
$$|h_1| < \delta_2 \implies \big| u(x_0+h_1, y_0) - u(x_0,y_0) - h_1 u_x(x_0,y_0) \big| < \frac{\varepsilon}{6} |h_1|$$
$$|h_1| < \delta_3 \implies \big| u_y(x_0+h_1, y_0) - u_y(x_0,y_0) \big| < \frac{\varepsilon}{6}$$
(since u_y is continuous)

$\to$ do the same for terms containing v and obtain $\delta_4, \delta_5, \delta_6$

Take δ = minimum of $\delta_1 - \delta_6$.

Then for $0 < |h| < \delta$, we have ($|h_1|$ and $|h_2| \le |h| < \delta$) ⑧

$$\text{L.H.S.} \;\not\leq\; \frac{1}{|h|} \; 6 \; \frac{1}{6} \varepsilon |h| \;=\; \varepsilon \quad \text{as required.}$$

□

(1.5) Examples

(i) $f = \underbrace{x^2 + y^2}_{u} + i(\underbrace{3x + y}_{v})$ is not $\mathbb{C}$-differentiable since

$$u_x = 2x \qquad v_x = 3$$
$$u_y = 2y \qquad v_y = 1$$

and C-R equations do not hold.

(ii) $f = \underbrace{x^2 - y^2}_{u} + i(\underbrace{2xy}_{v})$ is $\mathbb{C}$-differentiable since

$$\left.\begin{array}{ll} u_x = 2x & v_x = 2y \\ u_y = -2y & v_y = 2x \end{array}\right\} \Rightarrow \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \quad \checkmark$$

In fact, in this case $f(z) = z^2$ $\Big(= (x + iy)^2$
$= x^2 - y^2 + 2xy\, i \Big)$