Lecture 10

(10.0) Recall that last time we proved the following theorem

Identity Theorem: Let $D \subset \mathbb{C}$ be a connected open set and $f, g : D \to \mathbb{C}$ two holomorphic functions. If there exist $a_n \in D$ $(n = 1, 2, 3, \ldots)$ and $a \in D$ such that $\lim_{n\to\infty} a_n = a$ and $f(a_n) = g(a_n)$ for every n, then $f = g$ on D.

This theorem was a direct application of power series expansion of a holomorphic function.

Now we can investigate the behavior of a function on a punctured domain.

(10.1) Let $D \subset \mathbb{C}$ be a non-empty (connected) open set and $a \in D$. Assume that $f : D \setminus \{a\} \to \mathbb{C}$ is a holomorphic function. There are three possibilities:

I. $\lim_{z\to a} (z-a) f(z) = 0$.

In this case we can extend f to the domain D.

Proof. Set $g(z) = (z-a)^2 f(z)$. Then $g(z)$ is holomorphic on D, by setting $g(a) = 0$. This is clear, since

$$g'(a) = \lim_{z\to a} \frac{g(z) - g(a)}{z-a} = \lim_{z\to a} \frac{g(z)}{z-a} \quad (\text{as } g(a) = 0)$$

$$= \lim_{z\to a} (z-a) f(z) \text{ exists and is equal to } 0.$$

Hence $g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ with $c_0 = c_1 = 0$

$= (z-a)^2 \sum_{n=2}^{\infty} c_n (z-a)^{n-2}$ on $D_r(a)$ = open disc of radius r centered at a.

So $f(z) = \sum_{n=2}^{\infty} c_n (z-a)^{n-2}$ and therefore f can be extended to the domain D by setting $f(a) = c_2$.

II. There exists $N \geq 1$ such that $\lim_{z \to a} (z-a)^N f(z)$ exists and $\neq 0$.

In this case, set $g(z) = (z-a)^N f(z)$ $(z \neq a)$. Then $\lim_{z\to a} (z-a) g(z) = 0$

$\left(g(a) = \lim_{z \to a} (z-a)^N f(z) \right)$

and by previous part $g(z)$ extends to the domain D.

Write $g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ on an open disc of radius $r > 0$ centered at a ($D_r(a)$)

Then $f(z) = \dfrac{g(z)}{(z-a)^N} = \sum_{n=0}^{\infty} c_n (z-a)^{n-N} = \sum_{\ell=-N}^{\infty} c_{\ell+N} (z-a)^{\ell}$

III For every $N \geq 1$, $\lim_{z \to a} (z-a)^N f(z)$ does not exist.

The point $a \in D$ is called singularity of f.

I. a is called a removable singularity

II. a is called a pole (of order N)

III. a is called an essential singularity.

(10.2) Examples

(1) $f(z) = \dfrac{\sin(z)}{z}$ for $z \in \mathbb{C} - \{0\}$.

$\lim_{z \to 0} z f(z) = \sin(0) = 0$. So 0 is a removable singularity

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\Rightarrow f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} \cdots$$ Taylor Series expansion near 0

set $f(0) = 1$. This defines f on $\mathbb{C}$.

(2) $f(z) = \dfrac{z-1}{z^4}$ has a pole at $z=0$ of order 4, since

$\lim_{z \to 0} z^4 f(z) = \lim_{z \to 0} (z-1) = -1$ exists and is non-zero.

$$f(z) = -z^{-4} + z^{-3}$$

(3) $f(z) = e^{1/z}$ has essential singularity near $z = 0$.

(10.3) Alternate Characterization I

Theorem: Again let $D \subset \mathbb{C}$ be an open set, $a \in D$ and $f: D - \{a\} \to \mathbb{C}$ a holomorphic function

(i) a is a removable singularity if and only if there is $r > 0$ such that $|f(z)|$ is bounded on $\{z$ such that $0 < |z-a| < r\}$

(ie. ~~∄~~ there is $M > 0$ such that $|f(z)| \leq M$ for every z with $|z-a| < r$ $(z \neq a)$)

(ii) a is a pole if and only if $\lim_{z \to a} |f(z)| = \infty$.

(iii) a is an essential singularity if and only if

$\lim_{z \to a} |f(z)|$ does not exist.

Proof. (i) If a is removable then f extends to a holomorphic function on D. Therefore given ε (say $= 1$) there is $r > 0$ such that $0 < |z-a| < r$ implies $|f(z) - f(a)| < 1$. Hence $|f(z)|$ is bounded.

Conversely, if $|f(z)|$ is bounded on some $D_r(a) - \{a\}$ (this is open disc of radius r, centered at a, with a removed from it), then $\lim_{z \to a} (z-a) f(z) = 0$. Hence a is removable.

(ii) If a is a pole of order $N \geq 1$, then $f(z) = \dfrac{g(z)}{(z-a)^N}$ where $g(a) \neq 0$. Therefore $\lim_{z \to a} |f(z)| = \lim_{\substack{r \to 0^+ \\ (r \in \mathbb{R})}} \dfrac{|g(a)|}{r^N} = +\infty$.

Conversely, if $\lim_{z \to a} |f(z)| = \infty$, then there is $r > 0$ such that $|f(z)| \geq 1$ for every z with $0 < |z-a| < r$.

Set $h(z) = \dfrac{1}{f(z)}$. Then $|h(z)| \leq 1$ is bounded on $D_r(a) - \{a\}$

hence (by previous part) $h(z)$ extends to a holomorphic function on D. If $h(z)$ has a zero of order $N \geq 1$ at $z = a$ $\left(|h(a)| = \lim_{z \to a} |h(z)| = 0 \text{, so } h(a) = 0 \right)$

Then $h(z) = (z-a)^N g(z)$; $g(a) \neq 0$ and perhaps on a smaller disc $D_{r_1}(a)$, g is never zero. (see last lecture : Lecture 9 section (9.5)).

Hence $f(z) = \dfrac{g(z)^{-1}}{(z-a)^N}$ near a (on $D_{r_1}(a) - \{a\}$)

$\lim_{z \to a} (z-a)^N f(z) = \dfrac{1}{g(a)} \neq 0$. So a is a pole of f of order N.

(iii) is by exclusion ($\lim_{z \to a} |f(z)|$ has only 3 options : finite, infinite, or d.n.e.)

$\square$

(10.4) Laurent Series. (centered at 0, for convenience only).

Let $D \subset \mathbb{C}$ be open set, $0 \in D$ and $f : D - \{0\} \to \mathbb{C}$ a holomorphic function. Pick $r > 0$ so that $D_r(0) \subset D$.

Theorem. There exist c_n $(n \in \mathbb{Z})$ complex numbers such that

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n = \sum_{n=-\infty}^{\infty} c_n z^n \leftarrow \text{converges uniformly on } D_r(0) - \{0\}.$$

(Laurent Series of f centered at 0)

Moreover $c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$ (γ = circle of radius $0<\rho<r$)

(Note: by Cauchy's Theorem

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$ is independent of the radius $0<\rho<r$ of γ.)

Note: uniform convergence of a series $\sum_{n=-\infty}^{\infty} c_n z^n$ is defined similarly.

Namely, given a compact set K contained in $D_r(0) - \{0\}$ and $\varepsilon > 0$

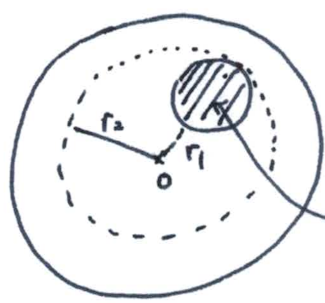
we can find $N > 0$ such that

$$\left.\begin{array}{l} |c_n z^n + \cdots + c_{n+p} z^{n+p}| < \varepsilon \\ |c_{-n} z^{-n} + \cdots + c_{-n-p} z^{-(n+p)}| < \varepsilon \end{array}\right\} \text{ for every } n \ge N,\ p \ge 0,\ z \in K$$

That is, both $\sum_{n=0}^{\infty} c_n z^n$ and $\sum_{n=-\infty}^{-1} c_n z^n$ converge uniformly.

Proof of Theorem.

Let K be a compact set contained in $D_r(0) - \{0\}$.

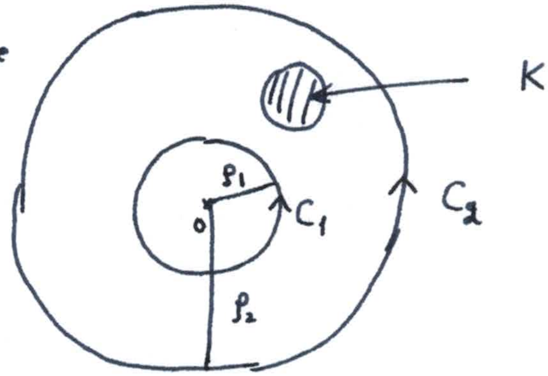


Let $0 < r_1 < r_2 \lneq r$ be such that

$r_1 \le |z| \le r_2$ for every $z \in K$.

Pick ρ_1 and ρ_2 : $0 < \rho_1 < r_1$
$r_2 < \rho_2 < r$

Let C_1 and C_2 be counterclockwise circles of radii ρ_1 and ρ_2 respectively.



For every $w \in K$ and $|z| = \rho_2$

$$\left|\frac{w}{z}\right| < 1$$

$$\Rightarrow \quad \frac{1}{z-w} = \frac{1}{z}\,\frac{1}{1-\frac{w}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{w^n}{z^n} \qquad \text{uniformly on } (|z| = \rho_2)$$

Similarly
$$\frac{1}{z-w} = -\frac{1}{w-z} = -\frac{1}{w}\left(\frac{1}{1-\frac{z}{w}}\right) = -\frac{1}{w}\sum_{n=0}^{\infty}\frac{z^n}{w^n} \qquad \text{uniformly on } (|z| = \rho_1)$$

Now by Cauchy's Integral formula

$$\int_{C_2}\frac{f(z)}{z-w}\,dz - \int_{C_1}\frac{f(z)}{z-w}\,dz = 2\pi i\, f(w)$$

$$\Rightarrow \quad f(w) = \frac{1}{2\pi i}\int_{C_2}\sum_{n=0}^{\infty} f(z)\,\frac{w^n}{z^{n+1}}\,dz - \frac{1}{2\pi i}\left(-\int_{C_1} f(z)\sum_{n=0}^{\infty}\frac{z^n}{w^{n+1}}\,dz\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z^{n+1}}\,dz\right) w^n + \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_1} f(z)\cdot z^n\,dz\right) w^{-n-1}$$

(by uniform convergence Theorem (7.3) of Lecture 7)

$\square$

(10.4) Alternate Characterization II.

Write the Laurent Series expansion of $f(z)$ centered at a.

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

I. $c_{-1} = c_{-2} = \cdots = 0$ Removable Singularity

II $c_{-N-1} = c_{-N-2} = \cdots = 0$ Pole of order N

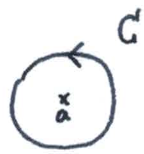
$c_{-N} \neq 0$

$(N \geq 1)$

III ~~$c_{-n} \neq 0$ for every $n = 1, 2, 3, \ldots$~~ Essential Singularity.

For every $n \geq 1$ there is $m > n$ such that $c_{-m} \neq 0$

(10.5) Define $\operatorname{Res}_{z=a} f(z)$ (called residue of f at a) as



$$\operatorname{Res}_{z=a} f(z) = \frac{1}{2\pi i} \int_C f(z)\, dz$$

If $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ then $\boxed{\operatorname{Res}_{z=a} f(z) = c_{-1}}$

(10.6) Examples

(1) Compute $\operatorname{Res}_{z=0} \dfrac{\sin(z)}{z^7}$.

$$\frac{\sin(z)}{z^7} = z^{-7}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots\right)$$

$$= z^{-6} - \frac{z^{-4}}{3!} + \frac{z^{-2}}{5!} - \frac{1}{7!} + \frac{z^2}{9!} - \cdots$$

$$\text{Coeff of } z^{-1} = 0 = \operatorname{Res}_{z=0} \frac{\sin(z)}{z^7}.$$

(II) Compute $\operatorname{Res}_{z=0} \dfrac{1}{e^z - e^{-z}}$

Solution 1. $$\frac{1}{e^z - e^{-z}} = \frac{1}{\left(1 + z + \frac{z^2}{2!} + \cdots\right) - \left(1 - z + \frac{z^2}{2!} - \cdots\right)}$$

$$= \frac{1}{2z} \cdot \frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}$$ does not vanish at $z=0$

$$\Rightarrow \operatorname{Res}_{z=0} \frac{1}{e^z - e^{-z}} = \frac{1}{2}$$

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Solution 2. $$\frac{1}{2\pi i} \int_C \frac{1}{e^z - e^{-z}}\, dz = \frac{1}{2\pi i} \int_C \frac{z}{e^z - e^{-z}} \frac{dz}{z} :$$

$$= \frac{1}{2} \quad \text{because}$$

$\lim_{z \to 0} \dfrac{z}{e^z - e^{-z}} = \dfrac{1}{2}$ (Check this) and by Cauchy's integral formula

$$\frac{1}{2\pi i} \int_C \frac{z}{e^z - e^{-z}} \frac{dz}{z} = \left[\frac{z}{e^z - e^{-z}}\right]_{\text{set } z=0}$$