Lecture 11

(11.0) Recall that last time we classified singularities of a function. For a holomorphic function $f : D - \{a\} \to \mathbb{C}$ (where $D \subset \mathbb{C}$ an open set and $a \in D$) we had the following three possibilities

Removable singularity : in this case any one of the following equivalent assertions is true

- f can be extended to a holomorphic function $D \to \mathbb{C}$.
- $\lim_{z \to a} (z-a) f(z) = 0$
- $|f(z)|$ is bounded in a disc around a (that is, there exists $M > 0$ and $r > 0$ such that $|f(z)| \leq M$ for every z with $0 < |z-a| < r$)
- If $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ is the Laurent series of f. then $c_{-1} = c_{-2} = c_{-3} = \dots = 0$.

Pole (of order $N \geq 1$) : in this case any one of the following equivalent assertions is true.

- $f(z) = \dfrac{g(z)}{(z-a)^N}$ where $g : D \to \mathbb{C}$ is holomorphic and $g(a) \neq 0$.
- $\lim_{z \to a} (z-a)^{N+1} f(z) = 0$ and N is smallest such positive integer

- $\lim_{z \to a} |f(z)| = \infty$
- If $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ is the Laurent series, then

$$c_{-N-1} = c_{-N-2} = \dots = 0$$

Essential singularity: in this case any of the following equivalent assertions is true

- $\lim_{z \to a} |f(z)|$ does not exist
- if $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ is the Laurent series of f, then

for any $n \geq 1$, there exists $N \geq n$ such that $c_{-N} \neq 0$.

$\operatorname{Res}_{z=a} f(z)$ is defined as $\frac{1}{2\pi i} \int_{\circlearrowleft a} f(z)\, dz$ and is equal to the coefficient c_{-1} in the Laurent Series expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ of f.

(11.1) Singularity at infinity.

Let $f : D \to \mathbb{C}$ be a holomorphic function on a domain for which there is $R > 0$ such that $|z| \geq R$ implies $z \in D$ (that is, D contains the unbounded set which is outside the circle of radius R centered at 0)

The type of singularity of f at $z = \infty$ (removable, pole or essential) is the type of singularity of $g(z) = f\left(\frac{1}{z}\right)$ at $z = 0$. Let us spell it out:

Removable case: In this case $\lim\limits_{z \to \infty} \frac{1}{z} f(z) = 0$ (i.e. given $\varepsilon > 0$, there exists $R > 0$ such that $\left| \frac{f(z)}{z} \right| < \varepsilon$ for every z with $|z| > R$)

Equivalently we can say that $|f(z)|$ is bounded on the (infinite - unbounded) set $\{z$ such that $|z| > R\}$.

(The fact that these statements are equivalent can be easily checked for $g(z) = f\left(\frac{1}{z}\right)$ near $z = 0$ - see section (10.3) of Lecture 10)

Pole of order $N \geq 1$. In this case $\lim\limits_{z \to \infty} z^{-N-1} f(z) = 0$ (and N is smallest such number). Equivalently $\lim\limits_{z \to \infty} |f(z)| = \infty$; or $f(z) = z^N h(z)$ where $h(z)$ has removable singularity at ∞ and "$h(\infty)$" $= \lim\limits_{z \to \infty} h(z) \neq 0$.

Essential singularity $\lim\limits_{z \to \infty} f(z)$ does not exist.

(11.2) Let f be an entire function. That is, f is holomorphic on the domain $= \mathbb{C}$.

<u>Theorem.</u> (1) f has a removable singularity at ∞ if and only if f is constant

(2) f has a pole of order $N \geq 1$ if and only if f is a polynomial of degree N.

<u>Proof</u> (1) was Liouville's Theorem. (Lecture 5 section (5.3)).

Namely: f has removable singularity at ∞ is same as saying that $|f(z)|$ is bounded on the set $|z| \geqslant R$. On the closed disc of radius R, $|f(z)|$ is bounded since $|f(z)|$ is continuous (so it must take an absolute maximum value on closed disc). Hence $|f(z)|$ is bounded for every $z \in \mathbb{C}$. By Liouville's Theorem, this implies that f is constant.

(2) The easiest proof of (2) is by using the inequality proved in Lecture 8, section (8.7) for coefficients of Taylor series of f.

Namely, let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be the Taylor series of f near $z=0$. Then

$$c_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z^{n+1}} \, dz$$

C_r = circle of radius $r > 0$ centered at 0.

If $M(r)$ = absolute max. of $|f(z)|$ on C_r then

$$|c_n| \leq \frac{1}{2\pi} \frac{M(r)}{r^{n+1}} 2\pi r = \frac{M(r)}{r^n}.$$

Now if $f(z)$ has a pole of order $N \geq 1$ at ∞, then $|z^{-N} f(z)|$ is bounded near ∞, i.e. there exists $R>0$ & $M>0$ such that

$$|z^{-N} f(z)| \leq M \quad \text{for every } |z| > R.$$ (and N is smallest such positive integer).

Therefore, for $r > R$, we get $|c_n| \leq \dfrac{M(r)}{r^n} \leq \dfrac{M r^N}{r^n} = \dfrac{M}{r^{n-N}}$

For $n > N$, $\dfrac{M}{r^{n-N}} \to 0$ as $r \to \infty$. Hence $|c_n| = 0$.

$$\Rightarrow \quad f(z) = c_0 + c_1 z + \dots + c_N z^N \quad \text{(poly. of degree } N)$$

$c_N = \lim_{z \to \infty} z^{-N} f(z)$ is assumed to be non-zero. (as N is the order of the pole)

$\square$

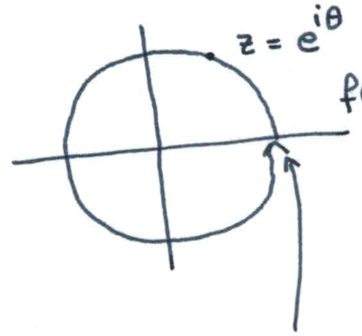
(11.3) Functions of the type z^a $(a \in \mathbb{C})$

For $a \in \mathbb{C}$, z^a is defined by: $z^a = e^{a \log(z)}$. This definition only makes sense (as a holomorphic function) once $\log(z)$

is properly defined. Thus in order to define $f(z) = z^a$, we need to specify a domain where it is defined, similar to $\log(z)$.

For instance, $f(z) = z^{\frac{1}{2}}$ cannot be defined as a holomorphic function on entire complex plane (since it will not be differentiable at 0). This is not the only problem though. We cannot define square-root as a holomorphic function on $\mathbb{C} - \{0\}$. The reason is analogous to that we saw for $\log(z)$.

Namely $z^{\frac{1}{2}} = e^{\frac{1}{2}\log(z)}$. The ambiguity in defining log will persist here. If we fix the value of $f(z) = z^{\frac{1}{2}}$ at 1 to be $f(1) = 1$ the upon circling around 0 once we will get $f(1) = -1$



$z = e^{i\theta}$ $0 \leq \theta \leq 2\pi$

$f(z) = e^{i\theta/2}$

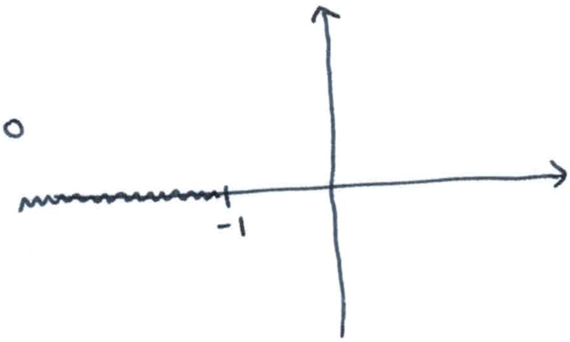
$\lim_{t \to 2\pi} f(e^{it}) = e^{\pi i} = -1 \neq 1$

Conclusion: for $a \in \mathbb{C}$, $a \notin \mathbb{Z}$,

$z^a = e^{a\log(z)}$ is only defined at $\mathbb{C} - \mathbb{R}_{\leq 0}$ by taking the corresponding holomorphic log (recall: $\log(z) = \ln|z| + i\arg(z)$ is a holomorphic function $: \mathbb{C} - \mathbb{R}_{\leq 0} \to \mathbb{C}$ $(\arg(z) \in (-\pi, \pi))$)

(11.4) Examples: (1) Let $a \in \mathbb{C}$ and consider $f(z) = (1+z)^a$ defined (as a holomorphic function) on the domain $D = \mathbb{C} - (-\infty, -1]$

$$f(z) = \sum_{n=0}^{\infty} \frac{a(a-1)\cdots(a-n+1)}{n!} z^n \quad \text{near } z=0$$



D

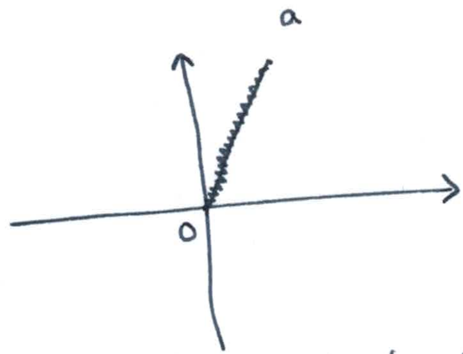
$$\frac{1}{n!}\left(\frac{d}{dz}\right)^n \cdot (1+z)^a = \frac{a(a-1)\cdots(a-n+1)}{n!}(1+z)^{a-n}$$

(2) $f(z) = \log(1 - az^{-1})$ is holomorphic function on the domain $\mathbb{C} - $ (straight line joining 0 and a)

$1 - az^{-1} = -t$ for a positive real number $t \geq 0$

$\equiv \quad 1 + t = az^{-1}$

$\equiv \quad z = \dfrac{a}{1+t}$



Domain of $\log(1 - az^{-1})$

So z is a (positive) real multiple of a and the positive real $\frac{1}{1+t}$ goes from 0 to 1 as t goes (down) from ∞ to 0.

(11.5) Non-isolated case.

The set up so far has been the following: $D \subset \mathbb{C}$ is an open set, $a \in D$ and $f : D - \{a\} \to \mathbb{C}$ a holomorphic function. Sometimes it is referred to as: 'a' is an isolated singularity of f.

Non-isolated case: Assume $a_1, a_2, a_3 \dots \in D$ and $a = \lim_{n\to\infty} a_n \in D$.

If $f : D - \{a, a_1, a_2, \dots\} \to \mathbb{C}$ is holomorphic and each a_n $(n = 1, 2, 3, \dots)$ is a singularity of f (not removable), then a is non-isolated singularity of f. In this case a cannot be removable or a pole.

Theorem. $\lim_{z\to a} |f(z)|$ does not exist

Proof. If $\lim_{z\to a} |f(z)|$ exists, then it is either finite or infinity.

Finite case: $|f(z)|$ will then be bounded in a disc around a. (see Theorem (10.3) of Lecture 10). But this disc contains infinitely many a_n's since $\lim_{n\to\infty} a_n = a$. This contradicts the assumption that each a_n is a singularity which is not removable

Infinite case: If $\lim_{z\to a} |f(z)| = \infty$. then follow the proof of (ii) of Theorem (10.3) of Lecture 10 to write $f(z) = \dfrac{g(z)}{(z-a)^N}$ where g is a holomorphic function on a disc near a $(g(a) \neq 0)$. This again leads to a contradiction since infinitely many a_n's must be in this disc

□