# Lecture 12



(12.0) Recall that we introduced the notion of (isolated) singularities and classified them into three types: removable, pole and essential.

Definition. A <u>meromorphic function</u> on a domain $D$ is a function (holomorphic) $f : D - A \to \mathbb{C}$ ($A \subset D$ is a subset) which has either removable singularity or a pole at each $a \in A$.

According to Theorem (11.5) of last lecture, the set $A$ (assuming it consists entirely of poles) cannot have a limit point. That is, we cannot have $a_1, a_2, a_3, \ldots \in A$ which converge to a point in $D$.

e.g. $f(z) = \operatorname{cosec}(z) = \dfrac{1}{\sin(z)}$ has poles (of order 1) at each $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Hence $\operatorname{cosec}(z)$ is a meromorphic function on the whole complex plane.

Poles of order 1 are often referred to as <u>simple poles</u>.

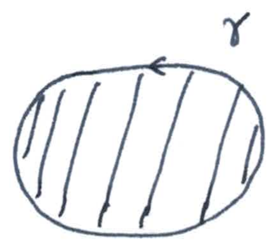
(12.1) Let $f(z)$ be a meromorphic function on $D \subset \mathbb{C}$ (open set) and $A \subset D$ be its set of poles. Consider a simple closed curve $\gamma : [a,b] \to D$ such that $\operatorname{Int}(\gamma) \subset D$; and $\gamma$ does not pass through any point of $A$.

<u>Theorem</u>.
(1) There are only finitely many poles in $\operatorname{Int}(\gamma)$.

(2) $$\frac{1}{2\pi i} \int_\gamma f(z)\,dz = \sum_{j=1}^{m} \operatorname*{Res}_{z=a_j} (f(z))$$

where $\{a_1, a_2, \dots, a_m\}$ are poles in $\mathrm{Int}(\gamma)$.

Proof. (1) Consider the closed and bounded subset $K \subset D$ consisting of $\mathrm{Int}(\gamma)$ and $\gamma$.

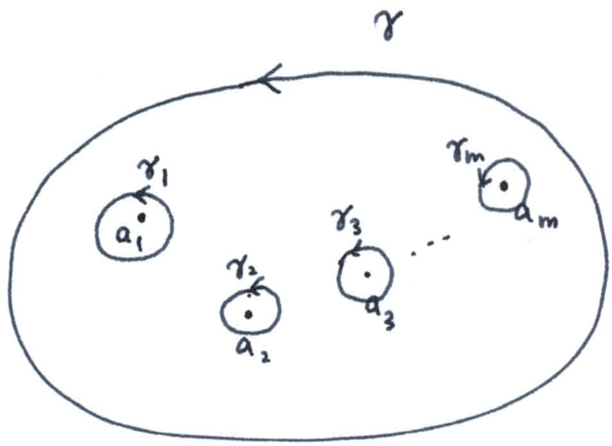



Compact set $K$

If there are infinitely many poles in $K$ then they must have a limit point, say $a^* \in K$. (see Theorem (A.3) of Optional Reading 1).

This limit point is still in $D$ (in fact in $\mathrm{Int}(\gamma)$ since $\gamma$ does not pass through any poles). But then $a^*$ is neither removable nor a pole (see Theorem (11.5) of Lecture 11) contradicting the definition of a meromorphic function.

(2) Now we can draw small contour $\gamma_j$ (counter clockwise always) around $a_j$ such that every $a_k$ $(k \neq j)$ is outside of $\gamma_j$.




By Cauchy's ~~std~~ theorem (see section (5.0) of Lecture 5)

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = \sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\gamma_j} f(z)\,dz$$

$$= \sum_{j=1}^{m} \mathrm{Res}_{z=a_j}(f(z)) \quad \text{by definition of residue}$$

$\square$

(12.2) On Homework 5 . problem (3) , residue is used to compute some definite integrals.

Example $\displaystyle\int_0^{2\pi} e^{\cos(\theta)} \cos(\theta - \sin(\theta))\, d\theta$ . Set $z = e^{i\theta}$ , so that

$$dz = ie^{i\theta} d\theta$$

$$\cos(\theta) = \frac{z + z^{-1}}{2} \quad \text{and} \quad \sin(\theta) = \frac{z - z^{-1}}{2i} \qquad d\theta = \frac{1}{i}\frac{dz}{z}$$

$$\cos(\theta - \sin(\theta)) = \cos(\theta)\cos(\sin(\theta)) + \sin(\theta)\sin(\sin(\theta))$$

$$= \cos(\theta)\cos\left(\frac{z - z^{-1}}{2i}\right) + \sin(\theta)\sin\left(\frac{z - z^{-1}}{2i}\right)$$

$$= \cos(\theta)\,\frac{e^{\frac{z - z^{-1}}{2}} + e^{\frac{-z + z^{-1}}{2}}}{2} + \sin(\theta)\,\frac{e^{\frac{z - z^{-1}}{2}} - e^{\frac{-z + z^{-1}}{2}}}{2i}$$

$$= \frac{1}{2}\left[\frac{z + z^{-1}}{2}\left(e^{\frac{z - z^{-1}}{2}} + e^{\frac{-z + z^{-1}}{2}}\right) - \frac{z - z^{-1}}{2}\left(e^{\frac{z - z^{-1}}{2}} - e^{\frac{-z + z^{-1}}{2}}\right)\right]$$

Therefore $e^{\cos\theta}\cos(\theta - \sin(\theta)) = e^{\frac{z + z^{-1}}{2}}\cos(\theta - \sin\theta)$

$$= \frac{1}{4}\left[(z + z^{-1})\left(e^{z} + e^{+z^{-1}}\right) - (z - z^{-1})\left(e^{z} - e^{+z^{-1}}\right)\right]$$

$$= \frac{1}{4}\left[2z\, e^{+z^{-1}} + 2\, z^{-1} e^{z}\right]$$

Hence we have to compute $\displaystyle\frac{1}{2i}\int_C \left(z\, e^{z^{-1}} + z^{-1} e^{z}\right)\frac{dz}{z}$

$C$ = circle of radius 1 centered at 0

$$= \pi \left( \frac{1}{2\pi i} \int_C \left(e^{z^{-1}} + z^{-2} e^{z}\right) dz \right)$$

$$e^{z^{-1}} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \cdots \quad \text{and} \quad z^{-2} e^{z} = z^{-2} + z^{-1} + \frac{1}{2!} + \frac{z}{3!} + \cdots$$

$$\text{Coeff of } z^{-1} \text{ in } e^{z^{-1}} + z^{-2} e^{z} = 2 = \frac{1}{2\pi i} \int_C \left(e^{z^{-1}} + z^{-2} e^{z}\right) dz$$

$$\Rightarrow \boxed{\text{Answer} = 2\pi}$$

(12.3) Examples of some infinite integrals.

Recall from Calculus II that for a function of real variable $h(x)$, the infinite integrals are understood to mean:

$$\int_a^\infty h(x)\,dx = \lim_{b \to \infty} \int_a^b h(x)\,dx$$

(if the limit exists, otherwise the integral is divergent)

Similarly

$$\int_{-\infty}^a h(x)\,dx = \lim_{b \to -\infty} \int_b^a h(x)\,dx$$

and

$$\int_{-\infty}^\infty h(x)\,dx = \int_{-\infty}^a h(x)\,dx + \int_a^\infty h(x)\,dx$$

(here $a \in \mathbb{R}$ is immaterial)

(both integrals must exist individually)

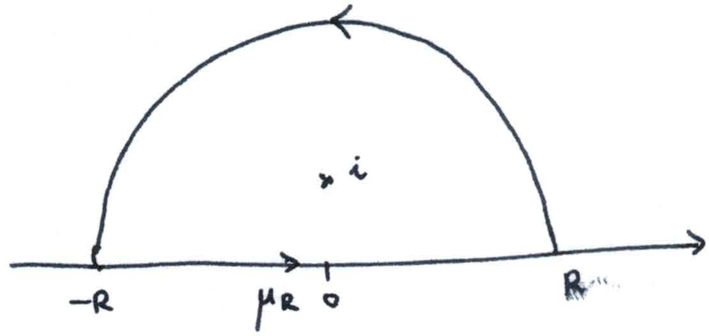
Example : Compute $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^3}$

Let $R>0$ be a real number. We need to compute $\displaystyle\lim_{R\to\infty}\int_0^R \frac{dx}{(x^2+1)^3}$.

Since the function is even, it is same to compute

$$\frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{(x^2+1)^3}$$

$\gamma_R(\theta) = Re^{i\theta}$ $(0 \le \theta \le \pi)$




Consider the integral

$$\frac{1}{2}\int_{C_R}\frac{dz}{(z^2+1)^3}$$ and assume $R>1$.

$\gamma_R$ = upper semicircle of radius $R$

$\mu_R = [-R, R]$

$C_R$ = simple closed curve $\gamma_R + \mu_R$

$z^2+1 = (z-i)(z+i)$

So $\dfrac{1}{(z^2+1)^3}$ has only one pole within $C_R$ (namely at $z=i$)

By Cauchy's integral formula

$$\frac{1}{2}\int_{C_R}\frac{dz}{(z^2+1)^3} = \frac{1}{2}\int_{C_R}\frac{1}{(z+i)^3}\frac{dz}{(z-i)^3} = \frac{1}{2}\,\frac{2\pi i}{2}\left(\frac{d}{dz}\right)^2\left(\frac{1}{(z+i)^3}\right)\Bigg|_{z=i}$$

$$= \frac{1}{2}\,\frac{2\pi i}{2}\left[\frac{12}{(z+i)^5}\right]_{\text{set } z=i} = \frac{1}{4}(2\pi i)\,\frac{12}{32\,i}$$

$$= \frac{3}{16}\pi$$

Hence we get $\frac{1}{2}\int_{\mu_R} \frac{dz}{(z^2+1)^3} + \frac{1}{2}\int_{\gamma_R} \frac{dz}{(z^2+1)^3} = \frac{3}{16}\pi$

$$\frac{1}{2}\int_{-R}^{R} \frac{dx}{(x^2+1)^3} - \frac{3}{16}\pi = -\frac{1}{2}\int_{\gamma_R} \frac{dz}{(z^2+1)^3}$$

Now $\left| \frac{-1}{2}\int_{\gamma_R} \frac{dz}{(z^2+1)^3} \right| \leq \frac{1}{2}\frac{\pi R}{(R^2-1)^3}$ $\downarrow$ as $R \to \infty$ $0$

( since $|z^2+1| \geqslant |z|^2 - 1 = R^2 - 1$, we get $\left|\frac{1}{z^2+1}\right| \leq \frac{1}{R^2-1}$

also length of $\gamma_R = \pi R$ )

So, given $\varepsilon > 0$, choose $R > 0$ large enough so that $\frac{1}{2}\frac{\pi R}{(R^2-1)^3} < \varepsilon$

Then $\left| \frac{1}{2}\int_{-R}^{R} \frac{dx}{(x^2+1)^3} - \frac{3}{16}\pi \right| \leq \frac{1}{2}\frac{\pi R}{(R^2-1)^3} < \varepsilon$

By definition of limit, this is same as saying that

$$\lim_{R\to\infty} \frac{1}{2}\int_{-R}^{R} \frac{dx}{(x^2+1)^3} = \frac{3}{16}\pi$$

(12.4) In examples of this type it becomes important to know whether $\left| \int_{\gamma_R} f(z)\, dz \right| \to 0$ as $R \to \infty$

A problem of this kind was given in Homework 2, problem (5) ( take the constant $A = 0$ there)

## Jordan's Lemma. 

Let $Q(z)$ be a holomorphic function on a domain which contains the subset $\{z \in \mathbb{C}$ such that $\operatorname{Im}(z) \geq 0$ and $|z| > c\} \subset \mathbb{C}$ (for a fixed $c>0$ real). Assume that $|Q(z)| \to 0$ as $|z| \to \infty$ and $\operatorname{Im}(z) \geq 0$. (That is, given $\varepsilon > 0$, there exists $R>0$ such that $|Q(z)| < \varepsilon$ for every $z$ for which $|z|>R$ and $\operatorname{Im}(z) \geq 0$). Then

$$\lim_{R \to \infty} \int_{\gamma_R} e^{miz} Q(z)\, dz = 0 \qquad (\text{for any } m>0 \text{ fixed})$$

Here $\gamma_R(\theta) = Re^{i\theta} \quad 0 \leq \theta \leq \pi$.

Proof. Given $\varepsilon > 0$, pick $R_0 > 0$ such that $|Q(z)| < \frac{m}{\pi}\varepsilon$ for every $z$ on $\gamma_R$ $(R \geq R_0)$. For any $R > R_0$, we get:

$$\left| \int_{\gamma_{R_0}} e^{miz} Q(z)\, dz \right| = \left| \int_0^\pi e^{miR(\cos\theta + i\sin\theta)} Q(Re^{i\theta})\, iRe^{i\theta}\, d\theta \right|$$

$$\leq \int_0^\pi \left| e^{miR\cos\theta} \right| \left| Q(Re^{i\theta}) \right| R \, \cancel{d\theta}\, e^{-mR\sin\theta}\, d\theta$$

$$\leq \frac{m\varepsilon}{\pi} \int_0^\pi R e^{-mR\sin(\theta)}\, d\theta = \frac{2m\varepsilon}{\pi} \int_0^{\pi/2} R e^{-mR\sin(\theta)}\, d\theta$$

$\left( \text{since } \left| e^{miR\cos\theta} \right| = 1 \right)$

Now $\sin(\theta) \geq \frac{2\theta}{\pi}$ when $0 \leq \theta \leq \frac{\pi}{2}$ $(*)$ Check this.

So we get $\left| \int_{\gamma_R} e^{miz} Q(z)\, dz \right| \leq \frac{2m\varepsilon}{\pi} \int_0^{\pi/2} R\, e^{-\frac{2mR}{\pi}\theta} \, d\theta$

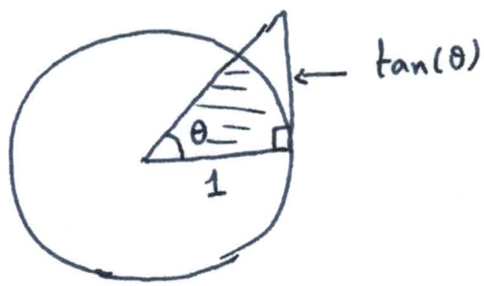
$$= \frac{2m\varepsilon}{\pi} R \frac{\pi}{2mR} \left[ -e^{-\frac{2mR\theta}{\pi}} \right]_0^{\pi/2}$$

$$= \varepsilon \left(1 - e^{-mR}\right) < \varepsilon \quad \text{as required.}$$

Proof of inequality $(*)$ $\sin(\theta) \geq \frac{2\theta}{\pi}$. for $0 \leq \theta \leq \frac{\pi}{2}$.

Recall that $\tan(\theta) \geq \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. This is proved by comparing area of the circular slice with area of the triangle :

$$\frac{\theta (1)^2}{\cancel{2}} \leq \frac{1}{\cancel{2}} \tan(\theta)$$




This implies that

$$\frac{d}{d\theta}\left(\frac{\sin(\theta)}{\theta}\right) = \frac{\cos(\theta)}{\theta} - \frac{\sin\theta}{\theta^2} = \frac{\cos\theta}{\theta^2}\left(\theta - \tan\theta\right) < 0$$

$\theta \in [0, \frac{\pi}{2}]$

So $\frac{\sin(\theta)}{\theta}$ is a decreasing function on $[0, \frac{\pi}{2}]$ and hence

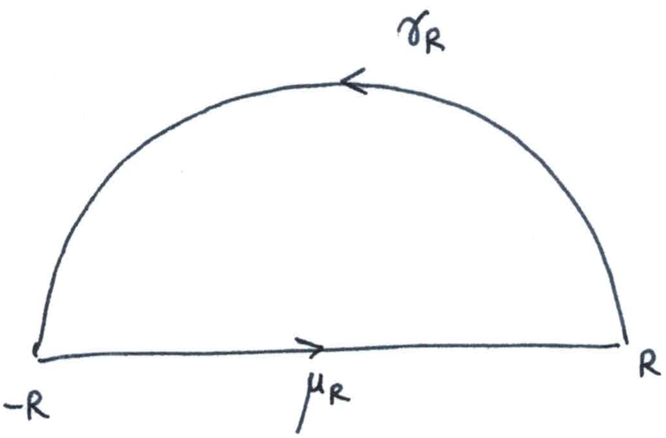
$$\frac{\sin(\theta)}{\theta} \geq \frac{\sin(\frac{\pi}{2})}{\pi/2} = \frac{2}{\pi} \quad \text{for } \theta \in [0, \tfrac{\pi}{2}].$$

$\square$

Ex. Use Jordan's lemma to prove that $\displaystyle\int_0^\infty \frac{\cos(x)}{x^2+a^2}\,dx = \frac{\pi}{2a}e^{-a}$

where $a>0$ is a real number.

Proof.



$$\int_{C_R} \frac{e^{iz}}{z^2+a^2}\,dz$$

$$= 2\pi i \frac{e^{i(ai)}}{ai+ai} \quad \text{(by Cauchy's formula)}$$

$C_R = \mu_R + \gamma_R$.

$$= \frac{\pi}{a}e^{-a}.$$

Now $\displaystyle\int_{\mu_R} \frac{e^{iz}}{z^2+a^2}\,dz = \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,dx = \int_0^R \frac{e^{ix}+e^{-ix}}{x^2+a^2}\,dx = 2\int_0^R \frac{\cos(x)}{x^2+a^2}\,dx$

and $\displaystyle\int_{\gamma_R} \frac{e^{iz}}{z^2+a^2}\,dz \to 0$ as $R\to\infty$ (by Jordan's Lemma)

$$\Rightarrow \quad 2\lim_{R\to\infty}\int_0^R \frac{\cos(x)}{x^2+a^2}\,dx = \frac{\pi}{a}e^{-a} \quad \text{as required.}$$

□