Lecture 13

(13.0) Recall : we have been studying the application of Cauchy's results on complex integration to real infinite integrals.

Example $$\int_{-\infty}^{\infty} \frac{x^4}{(a+bx^2)^4}\,dx = \frac{\pi}{16\, a^{3/2}\, b^{5/2}}$$ $a>0$ and $b>0$ are real numbers

$$\int_{-\infty}^{\infty} \frac{x^4}{(a+bx^2)^4}\,dx = \frac{1}{b^4}\int_{-\infty}^{\infty} \frac{x^4}{(x^2+c)^4}\,dx$$ where $c = \frac{a}{b} > 0$ real.

Since the function is even we can take $\frac{1}{b^4}\lim_{R\to\infty}\int_{-R}^{R} \frac{x^4}{(x^2+c)^4}\,dx$ as $\frac{1}{b^4}\int_{-\infty}^{\infty}\frac{x^4}{(x^2+c)^4}\,dx$. Reason is as follows:

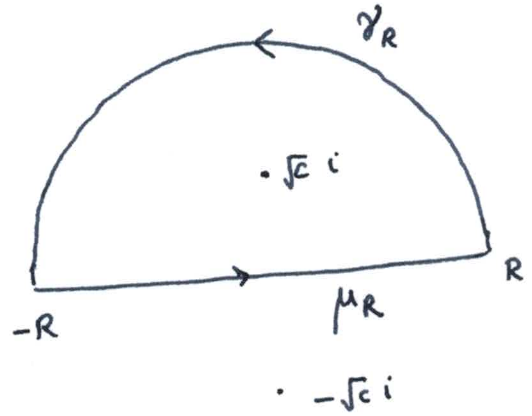
$$\frac{1}{b^4}\int_{-\infty}^{\infty}\frac{x^4}{(x^2+c)^4}\,dx = \frac{1}{b^4}\lim_{\substack{R\to\infty\\ S\to\infty}}\int_{-S}^{R}\frac{x^4}{(x^2+c)^4}\,dx$$

$$= \frac{1}{b^4}\lim_{\substack{R\to\infty\\ S\to\infty}}\left\{\frac{1}{2}\left[\int_{-S}^{0}\frac{x^4}{(x^2+c)^4}\,dx + \int_{0}^{S}\frac{x^4}{(x^2+c)^4}\,dx\right] + \frac{1}{2}\left[\int_{0}^{R}\frac{x^4}{(x^2+c)^4}\,dx + \int_{-R}^{0}\frac{x^4}{(x^2+c)^4}\,dx\right]\right\}$$

$$= \frac{1}{b^4}\lim_{T\to\infty}\int_{-T}^{T}\frac{x^4}{(x^2+c)^4}\,dx.$$

Now consider the complex integral

$$\int_{C_R} \frac{z^4}{(z^2+c)^4}\, dz$$



$C_R = \gamma_R + \mu_R$

$$\underset{\text{by Cauchy's formula}}{=} \frac{2\pi i}{3!} \left[\left(\frac{d}{dz}\right)^3 \frac{z^4}{(z+\sqrt{c}\, i)^4} \right]_{\text{set } z = \sqrt{c}\, i}$$

[long but direct computation]

$$= \frac{2\pi i}{6} \left(\frac{-24}{128} \frac{1}{(\sqrt{c}\, i)^3} \right) = \frac{\pi}{16} \frac{1}{c^{3/2}}.$$

Then we check that $\displaystyle\int_{\gamma_R} \frac{z^4}{(z^2+c)^4}\, dz \to 0$ as $R \to \infty$:

$$\left| \int_{\gamma_R} \frac{z^4}{(z^2+c)^4}\, dz \right| \leq \frac{R^4}{(R^2-|c|)^4}\, 2\pi R \to 0 \text{ as } R \to \infty.$$

$(c = a b^{-1})$

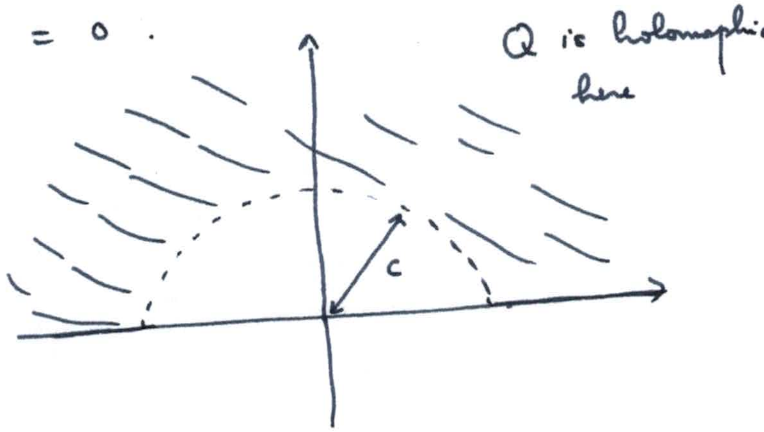
Hence
$$\lim_{R\to\infty} \frac{1}{b^4} \int_{\mu_R} \frac{z^4}{(z^2+c)^4}\, dz = \frac{1}{b^4} \frac{\pi}{16} \frac{1}{a^{3/2} b^{-3/2}}$$
$$= \frac{\pi}{16} \frac{1}{a^{3/2} b^{5/2}}.$$

(13.1) Also recall that we stated Jordan's Lemma :

Assume $Q(z)$ is a holomorphic function on a domain D which contains the set $\{z \in \mathbb{C}$ such that $\operatorname{Im}(z) \geq 0,\ |z| > c\}$ (see fig below) for some positive real number c. Assume that $\lim_{z \to \infty} |Q(z)| = 0$.

Then for every $m > 0$ real, we have

$$\lim_{R \to \infty} \int_{\gamma_R} e^{miz} Q(z)\, dz = 0 .$$

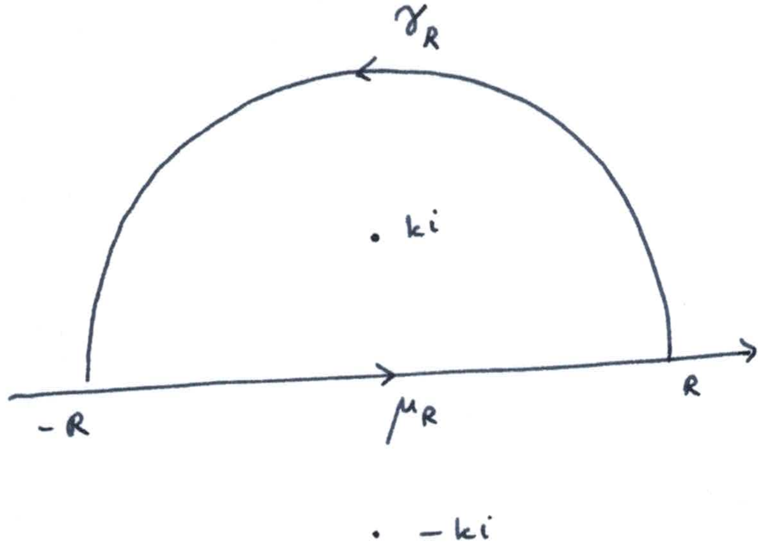


Example Let $k > 0$ and $a > 0$. Then

$$\int_0^\infty \frac{x \sin(ax)}{x^2 + k^2}\, dx = \frac{1}{2} \pi e^{-ka} .$$

Take $Q(z) = \dfrac{z}{z^2 + k^2}$, $c > k$ and $m = a$ to get

$$\lim_{R \to \infty} \int_{\gamma_R} \frac{e^{iaz} z}{z^2 + k^2}\, dz = 0 .$$



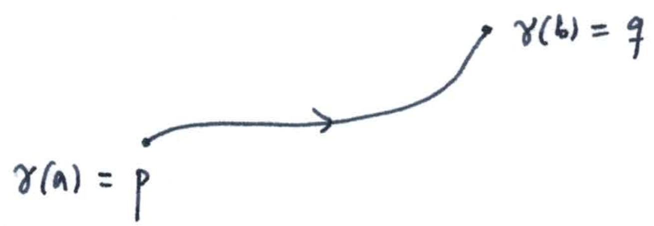
By Cauchy's formula: $$\int_{C_R} \frac{e^{iaz}\, z}{z^2+k^2}\, dz = 2\pi i \left[\frac{e^{iaz}\, z}{z+ki}\right]_{\text{set } z=ki}$$

$$= 2\pi i \, \frac{e^{aki^2}\, ki}{2ki} = \pi i \, e^{-ak}$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iax}\, x}{x^2+k^2}\, dx = \pi i \, e^{-ak}.$$ The imaginary part of the integral is $$\lim_{R\to\infty} \int_{-R}^{R} \frac{x \sin(ax)}{x^2+k^2}\, dx = \pi e^{-ak}.$$ Since the function $\frac{x \sin(ax)}{x^2+k^2}$ is even, we get $$\int_{0}^{\infty} \frac{x \sin(ax)}{x^2+k^2}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin(ax)}{x^2+k^2}\, dx = \frac{\pi}{2} e^{-ak}.$$

(13.2) A few remarks about <u>improper integrals</u>: where function has a pole on the curve γ.

$\gamma : [a,b] \longrightarrow \mathbb{C}$



Case I : The pole is at ~~the~~ one of the end points (say p)

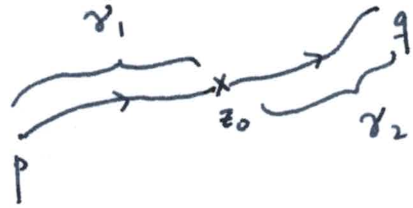
$$\int_{\gamma} f(z)\,dz \underset{\text{definition}}{=} \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\,dz$$

Disc of radius ε around p

(if exists!)

Case II : The pole is at some other point say z_0.

$\int_{\gamma} f(z)\,dz$ is defined as



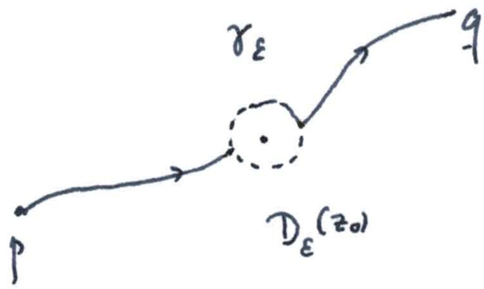
$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz \qquad \text{(if both of these exist)}$$

i.e.
$$\int_{\gamma} f(z)\,dz = \lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \int_{\gamma_1 : p \to z_0 - \varepsilon_1} f(z)\,dz + \int_{\gamma_2 : z_0 + \varepsilon_2 \to q} f(z)\,dz$$

Cauchy's Principal Value is defined as (for integrals $\int_{\gamma} f(z)\,dz$ when f has a pole on γ, but not at the end points)

$$\lim_{\varepsilon \to 0} \int_{\gamma_1 : p \to z_0 - \varepsilon} f(z)\,dz + \int_{\gamma_2 : z_0 + \varepsilon \to q} f(z)\,dz$$

In words, remove a disc of radius ε around z_0 and let γ_ε be the left over path. Then take the limit



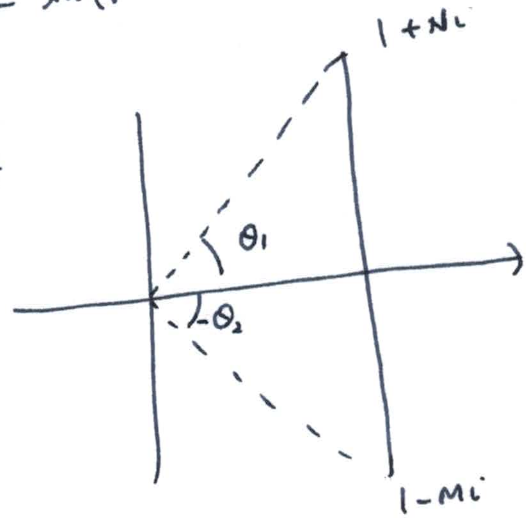
$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\, dz$$

Remark. Cauchy's P.V. may exist even though the improper integral does not.

Example. Let $\gamma(t) = 1 + it \qquad -\infty < t < \infty$

$$\int_\gamma \frac{1}{z}\, dz = \lim_{M,N \to \infty} \int_{1-Mi}^{1+Ni} \frac{1}{z}\, dz = \lim_{M,N\to\infty} \left(\log(1+Ni) - \log(1+Mi) \right)$$

$$= \lim_{M,N \to \infty} \ln\left(\sqrt{1+N^2}\right) - \ln\left(\sqrt{1+M^2}\right) + i\theta_1 + i\theta_2$$



Now $\lim\limits_{M,N\to\infty} \frac{1}{2}\left(\ln(1+N^2) - \ln(1+M^2) \right)$ does not exist (it depends on how M and N are related)

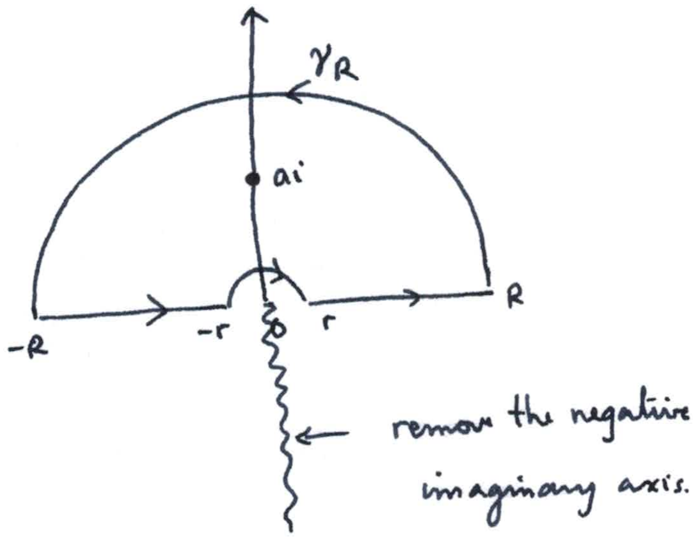
But $\displaystyle \text{P.V} \int_\gamma \frac{1}{z}\, dz = \lim_{N \to \infty} \int_{1-Ni}^{1+Ni} \frac{1}{z}\, dz = i\pi$ exists.

(13.3) Another example $\displaystyle\int_0^\infty \frac{\ln(x)}{x^2+a^2}\,dx$ $\quad$ ($a>0$ real)

Take logarithm to be defined on $\mathbb{C} - \mathbb{R}_{\leq 0}\cdot i$. Let us call it $\lg(z) = \ln|z| + i\arg(z)$

$\left(\frac{-\pi}{2} < \arg(z) < \frac{3\pi}{2} \right)$

$z \in \mathbb{C}$ such that $z \notin \mathbb{R}_{\leq 0}\cdot i$



Contour $C_{r,R}$

Step 1 : Cauchy's formula

$$\int_{C_{r,R}} \frac{\lg(z)}{z^2+a^2}\,dz = 2\pi i \left[\frac{\lg(z)}{z+ai} \right]_{\text{set } z = ai}$$

$$= \frac{2\pi i}{2ai}\left(\ln(a) + \frac{\pi}{2} i \right) = \frac{\pi}{a}\left(\ln(a) + \frac{\pi}{2} i \right)$$

Step 2 $\displaystyle\int_{\gamma_R} \frac{\lg(z)}{z^2+a^2}\,dz$. On γ_R : $z = Re^{i\theta}$ $(0 \leq \theta \leq \pi)$

$$|\lg(z)| = |\ln R + i\theta|$$
$$= \sqrt{\ln R^2 + \theta^2}$$
$$< \sqrt{2}\,\ln(R)$$

(if $\ln(R) > \pi$)
$R > e^\pi$

$$|z^2 + a^2| < R^2 - a^2 < \frac{3R^2}{4} \quad \text{if } 2a < R$$

So $$\left| \int_{\gamma_R} \frac{\lg(z)}{z^2+a^2}\, dz \right| < \frac{\sqrt{2}}{3/4} \frac{\ln(R)}{R^2} \cdot \pi R = \frac{4\sqrt{2}}{3}\pi \frac{\ln(R)}{R}$$

Now $$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{x^{-1}}{1} \quad (\text{L'Hospital}) \quad = 0$$

So $$\lim_{R\to\infty} \int_{\gamma_R} \frac{\lg(z)}{z^2+a^2}\, dz = 0.$$

<u>Step 3.</u> $$\int_{\gamma_r} \frac{\lg(z)}{z^2+a^2}\, dz \qquad \gamma_r(\theta) = re^{i\theta} \quad (0 \le \theta \le \pi) \qquad \begin{array}{l} z = re^{i\theta} \\ dz = rie^{i\theta} d\theta \end{array}$$

$$\lim_{r\to 0} \int_{\gamma_r} \frac{\lg(z)}{z^2+a^2}\, dz = \lim_{r\to 0} \int_0^{\pi} \frac{\ln(r)+i\theta}{r^2 e^{2i\theta}+a^2}\, rie^{i\theta}\, d\theta = 0.$$

Check : $$\lim_{r\to 0} r\left(\frac{\ln(r)+i\theta}{r^2 e^{2i\theta}+a^2}\right)\cdot i = 0$$ regardless of what θ is.

Hence $$\int_{-R}^{-r} \frac{\lg(z)}{z^2+a^2}\, dz + \int_r^R \frac{\lg(z)}{z^2+a^2}\, dz = \frac{\pi}{a}\left(\ln(a) + \frac{\pi}{2} i\right) \quad (\star)$$ as $r\to 0$, $R\to\infty$

here $\lg(-x) = \ln(x) + \pi i$

Real part of $(\star)$: $$2\int_r^R \frac{\ln(x)}{x^2+a^2}\, dx = \frac{\pi}{a}\ln(a)$$ as $r\to 0$, $R\to\infty$. □