# Lecture 14

## Holomorphic functions defined by integrals.

(14.0) Aim : To study functions coming from integrals in the following sense.

$$\underbrace{\boxed{K(t,z)\ ,\ \gamma}}_{\text{Input}} \rightsquigarrow \underbrace{\boxed{f(z) \overset{\text{definition}}{=} \int_\gamma K(t,z)\,dt}}_{\text{Output}}$$

$\gamma$ is a simple curve in $\mathbb{C}$

$K(t,z)$ is a complex-valued function of $t \in \gamma$, $z \in D$

( $D \subset \mathbb{C}$ a non-empty open set)

We will begin by writing down some general criteria for $f(z)$ to be a holomorphic function and so that $f'(z) = \int_\gamma \frac{\partial K(t,z)}{\partial z}\,dt$

(14.1) Finite case : $\gamma : [a,b] \to \mathbb{C}$ is a finite curve (simple)

i.e. length of $\gamma$ is finite.

Theorem. Assume that the following assumptions hold for $K(t,z)$

(1) For fixed $z \in D$, $K(t,z)$ and $\frac{\partial}{\partial z}K(t,z)$ are continuous functions of $t \in \gamma$ (image of $\gamma$)

(2) For fixed $t \in \gamma$, $K(t,z)$ is holomorphic function of $z$.

(3) Continuity of $\frac{\partial K}{\partial z}(t,z)$ in $z \in D$ is uniform with respect to $t \in \gamma$. (Same for $K(t,z)$).

Then $f(z) = \int_\gamma K(t,z)\,dt$ is a holomorphic function of $z \in D$

and its derivative is given by $f'(z) = \int_\gamma \frac{\partial}{\partial z} K(t,z)\,dt$.

<u>Remarks on the hypotheses</u>: • Assumption number (2) is there to make sure $K(t,z)$ can be differentiated in variable $z$. Number (1) is to make sure $\int_\gamma K(t,z)\,dt$ and $\int_\gamma \frac{\partial}{\partial z} K(t,z)\,dt$ exist.

Meaning of assumption number (3): Recall that continuity of $K_z(t,z) = \frac{\partial}{\partial z} K(t,z)$ in $z$ variable means the following

$\lim_{\substack{h \to 0 \\ (h \in \mathbb{C})}} K_z(t,z+h) = K_z(t,z)$. That is, given $\varepsilon > 0$, there is $\delta > 0$ such that $|h| < \delta \Rightarrow |K_z(t,z+h) - K_z(t,z)| < \varepsilon$.

However this $\delta$ may depend on $t$. (and of course $\varepsilon$).

Assumption (3) is saying that we can choose one $\delta > 0$ which works for every $t \in \gamma$. That is, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$|h| < \delta \Rightarrow |K_z(t,z+h) - K_z(t,z)| < \varepsilon$$

for every $t \in \gamma$.

( Similarly $K(t,z)$'s continuity in $z$ is uniform with respect to $t$, Proof. Let us begin by proving that $f(z) = \int_\gamma K(t,z)\,dt$ is continuous. That is, we have to prove that given $\varepsilon > 0$, we can find $\delta > 0$ such that: $|h| < \delta \Rightarrow |f(z+h) - f(z)| < \varepsilon$.

But $|f(z+h) - f(z)| = \left| \int_\gamma (K(t,z+h) - K(t,z))\,dt \right|$. So pick $\delta > 0$ (using assumption (3)) so that $|K(t,z+h) - K(t,z)| < \dfrac{\varepsilon}{\text{length of } \gamma}$ for every $t \in \gamma$. Then for $|h| < \delta$, we get

$$|f(z+h) - f(z)| = \left| \int_\gamma K(t,z+h) - K(t,z)\,dt \right| < \frac{\varepsilon}{\text{Length of } \gamma} \cdot \text{length of } \gamma$$

$$= \varepsilon.$$

as required.

Now we will prove that $\lim\limits_{h\to 0} \dfrac{f(z+h) - f(z)}{h}$ exists and equals $\int_\gamma K_z(t,z)\,dt$ (recall $K_z(t,z)$ is short hand for $\frac{\partial}{\partial z} K(t,z)$)

$$\frac{f(z+h) - f(z)}{h} = \int_\gamma \frac{K(t,z+h) - K(t,z)}{h}\,dt$$

By mean value theorem, there is a number $0 \le a \le 1$ so that

$$\frac{K(t,z+h) - K(t,z)}{h} = K_z(t, z+ah)$$

By continuity of $K_z(t,z)$, given $\varepsilon > 0$, we can find $\delta > 0$
(assumption (3))

such that $|p| < \delta \Rightarrow \left| K_z(t, z+p) - K_z(t,z) \right| < \dfrac{\varepsilon}{\text{length of } \gamma}$

For this $\delta$, we get that if $|h| < \delta$ then

$$\left| \frac{f(z+h) - f(z)}{h} - \int_\gamma K_z(t,z)\,dt \right| = \left| \int_\gamma K_z(t, z+ah) - K_z(t,z)\,dt \right|$$

$$< \frac{\varepsilon}{\text{length of } \gamma} \cdot \text{length of } \gamma = \varepsilon \quad \text{as required.} \quad \square$$

(14.2) Infinite case: $\gamma = [0, \infty) \subset \mathbb{C}$. That is,

$$f(z) = \int_0^\infty K(t,z)\,dt .$$

$K(t,z)$ is a complex-valued function of $t \in \mathbb{R}$ $(t \geq 0)$ and $z \in D$ (some domain in $\mathbb{C}$).

Theorem. Assume the following hypotheses

(1) For fixed $t$, $K(t,z)$ is a holomorphic function of $z$.

(2) $K(t,z)$ and $K_z(t,z)$ are continuous functions of both $t$ and $z$

(3) $\displaystyle\int_0^\infty K(t,z)\,dt$ and $\displaystyle\int_0^\infty K_z(t,z)$ exist and their convergence is uniform in $z \in D$.

Then $f(z) = \int_0^\infty K(t,z)\,dt$ is a holomorphic function

and $f'(z) = \int_0^\infty K_z(t,z)\,dt$

Remarks on the hypotheses :

- Number (3) : Since $\int_0^\infty g(t,z)\,dt = \lim_{R\to\infty} \int_0^R g(t,z)\,dt$, its existence is equivalent to the assertion that for a given $\varepsilon > 0$ we can find $R > 0$ such that $\left| \int_R^S g(t,z)\,dt \right| < \varepsilon$; for every $S > R$.

The assumption(s) on $K(t,z)$ and $K_z(t,z)$ are saying that this number $R > 0$ can be picked, once and for all, for every $z \in D$.

This last condition can be relaxed to uniformity of

$$\lim_{R\to\infty} \int_0^R K(t,z)\,dt \quad (\text{and similar one for } K_z(t,z))$$

on every closed and bounded set $K$ contained in $D$
(also called compact)

( see Lecture 7 sections (7.2) and (7.3) )

- <u>Number (2)</u> : Asserting that a function $g(x,y)$ is a continuous function of both $x$ and $y$ means that

$$\lim_{(x,y)\to(a,b)} g(x,y) = g(a,b)$$ which is a stronger assumption

than saying for fixed $x$, $g(x,y)$ is continuous in $y$ and for fixed $y$, $g(x,y)$ is continuous in $x$. For example (from Calculus III)

$$g(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous function of $y$ if $x$ is fixed and continuous function of $x$ if $y$ is fixed

but $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

This assumption (number (2)) is needed in order to be able to switch the order of integration

$$\int_{\mu} \left( \int_0^R K_z(t,z)\, dt \right) dz = \int_0^R \left( \int_{\mu} K_z(t,z)\, dz \right) dt$$

a result known as Fubini's Theorem. Its proof will be given in Optional Reading 2 - Appendix B).

<u>Proof</u>. The precise argument will be given in Optional Reading II

Here is the idea: for $w \in D$, pick a point $A \in D$ and a path $\mu$ from $A$ to $w$.


$$\int_\mu \left( \int_0^\infty K_z(t,z)\,dt \right) dz$$

$$= \int_0^\infty \left( \int_\mu K_z(t,z)\,dz \right) dt$$ (The flip of the order of integration is justified by assumptions (2) (3))

$$= \int_0^\infty \left( K(t,w) - K(t,A) \right) dt$$ (Fundamental Theorem of Calculus)

$$= f(w) - f(A).$$ (by definition of $f$)

So $$\frac{d}{dw} f(w) = \frac{d}{dw}\left( f(w) - f(A) \right) = \frac{d}{dw} \int_\mu \left( \int_0^\infty K_z(t,z)\,dt \right) dz$$

$$= \int_0^\infty K_z(t,w)\,dt$$ (again by fundamental theorem).

(see also Lecture 4, page 4,5) $\square$

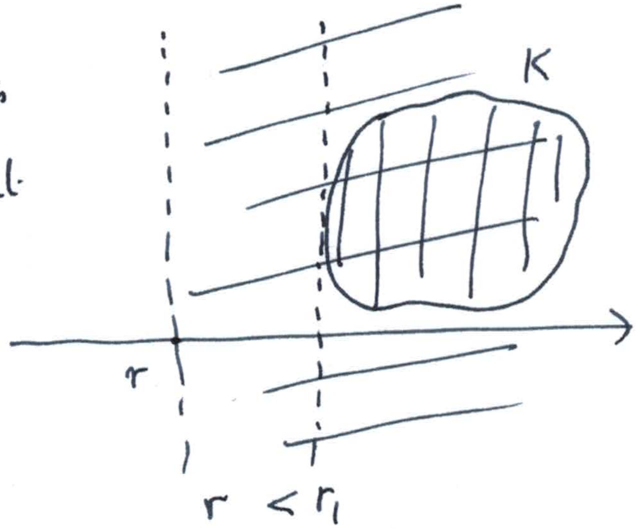
(14.3) Example: Laplace Transform.

Given $\varphi(t)$ a complex valued function of $t \in \mathbb{R}_{\geq 0}$.

$$(\mathcal{L}\varphi)(z) := \int_0^\infty \varphi(t)\, e^{-zt}\, dt$$ (Laplace transform of $\varphi$).

If there exist a real number $r$ and a positive number $K$ such that $|\varphi(t)| < K \cdot e^{rt}$, then $(\mathcal{L}\varphi)(z)$ defines a holomorphic function on the half plane $Re(z) > r$.

Let $K$ be a compact set in this domain. Pick $r_1 > r$ such that $Re(z) \geq r_1$ for every $z \in K$.




Then $$\left|\varphi(t)\, e^{-zt}\right| < K\, e^{-(r_1-r)t}$$

and $$\int_0^\infty e^{-(r_1-r)t}\, dt = \frac{1}{r_1-r}$$ exists.

Similarly $$\int_0^\infty \frac{\partial}{\partial z}\left(\varphi(t)\, e^{-zt}\right) dt$$ converges uniformly in $z \in K$

( same argument as before and check that $$\int_0^\infty t\, e^{-(r_1-r)t}\, dt = \frac{1}{(r_1-r)^2}$$ is still finite )

Thus we have checked hypothesis (3) of Theorem ((4.2). The other hypotheses ((1) and (2)) are absolutely trivial to verify.

Hence (by Theorem (14.2)) $\int_0^\infty \varphi(t)\, e^{-zt}\, dt$ defines a holomorphic function.

e.g. • $\varphi(t) = 1 \Rightarrow (\mathcal{L}\varphi)(z) = \int_0^\infty e^{-zt}\, dt = \left[\frac{e^{-zt}}{-z}\right]_0^\infty$

$\lim_{t\to\infty} e^{-zt} = 0$ if and only if $Re(z) > 0$

(d.n.e if $Re(z) = 0$)

So $(\mathcal{L}\varphi)(z) = \frac{1}{z}$ for $Re(z) > 0$.

• $\varphi(t) = e^t \Rightarrow (\mathcal{L}\varphi)(z) = \int_0^\infty e^{(1-z)t}\, dt = \frac{1}{z-1}$

for $Re(z) > 1$.

Ex. $\varphi(t) = \frac{t^n}{n!} \Rightarrow (\mathcal{L}\varphi)(z) = z^{-n-1}$ for $Re(z) > 0$.

$(n \geq 0)$