Lecture 15

(15.0) Recall : last time we listed hypotheses on a function $K(t,z)$ which imply that $\int_0^\infty K(t,z)\,dt$ defines a holomorphic function with its derivative given by $\int_0^\infty \frac{\partial K}{\partial z}(t,z)\,dt$.

Here $K(t,z)$ is a complex valued function of $t \in \mathbb{R}$ $(t \geq 0)$ and $z \in D \subset \mathbb{C}$ (D is an open subset), such that

(1) For fixed t, $K(t,z)$ is a holomorphic function of z.

(2) $K(t,z)$ and $\frac{\partial K(t,z)}{\partial z}$ are continuous functions of both t and z.

(3) $\int_0^\infty K(t,z)\,dt$ and $\int_0^\infty \frac{\partial K}{\partial z}(t,z)\,dt$ exist and their convergence is uniform (on closed bounded subsets A contained in D).

Then $f(z) := \int_0^\infty K(t,z)\,dt$ is a holomorphic function of $z \in D$

and $f'(z) = \int_0^\infty \frac{\partial K}{\partial z}(t,z)\,dt$

As an example, we introduced the Laplace transform of a function $\varphi(t)$: $(\mathcal{L}\varphi)(z) = \int_0^\infty \varphi(t)\, e^{-zt}\,dt$

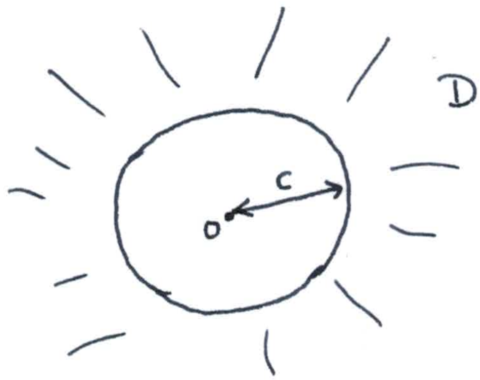
If $|\varphi(t)| < C \cdot e^{rt}$, then we checked the hypotheses for

for $D = \{ z \in \mathbb{C} \ : \ \mathrm{Re}(z) > r \}$

(15.1) An example with finite γ (see Theorem 14.1).

Let $g(w)$ be a holomorphic function on a domain which is outside of a disc centered around 0,

$$D = \{ w \in \mathbb{C} \ : \ |w| > c \} \quad (c \in \mathbb{R},\ c > 0)$$



Assume that

$$\lim_{w \to \infty} |g(w)| = 0$$

Borel Tranform of $g(w)$ is defined as

$$(Bg)(z) = \frac{1}{2\pi i} \int_{\gamma_R} g(w) e^{zw} \, dw$$

$\gamma_R(\theta) = R e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ for some $R > c$

e.g. $g(w) = \frac{1}{w}$

$$(Bg)(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{1}{w} e^{zw} \, dw = \left[e^{zw} \right]_{\text{set } w=0} = 1$$

by Cauchy's formula

e.g. $g(w) = w^{-n-1} \Rightarrow (Bg)(z) = \frac{z^n}{n!}$

(again by Cauchy's formula)

The hypotheses of Theorem (14.1) are easy to verify defining $(Bg)(z)$ as a holomorphic function of $z \in \mathbb{C}$.

e.g. if $\lim_{w \to \infty} w\, g(w)$ exists and equals a, then

$(Bg)(0) = a$.

Proof. (see Problem 5 of HW #2). $(Bg)(0) = \frac{1}{2\pi i} \int_{\gamma_R} g(w)\, dw$

is independent of R. (as long as $R > c$) by Cauchy's theorem.

$$(Bg)(0) = \frac{1}{2\pi i} \int_0^{2\pi} g(Re^{i\theta}) R e^{i\theta} \cdot i\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} g(Re^{i\theta}) Re^{i\theta}\, d\theta$$

Given $\varepsilon > 0$, choose $R_0 > 0$ so that for every $R > R_0$

$$|g(Re^{i\theta}) Re^{i\theta} - a| < \varepsilon$$

Then $$|(Bg)(0) - a| = \left| \frac{1}{2\pi} \int_0^{2\pi} g(Re^{i\theta}) Re^{i\theta}\, d\theta - a \right|$$

$$= \left| \frac{1}{2\pi} \int_0^{2\pi} g(Re^{i\theta}) Re^{i\theta}\, d\theta - \frac{1}{2\pi} \int_0^{2\pi} a\, d\theta \right|$$

$$= \left| \frac{1}{2\pi} \int_0^{2\pi} \left(g(Re^{i\theta}) Re^{i\theta} - a\right) d\theta \right| < \frac{\varepsilon \cdot \cancel{2\pi}}{\cancel{2\pi}} = \varepsilon.$$

So $|Bg(0) - a| = 0 \Rightarrow Bg(0) = a$ $\square$

(15.2) An example of infinite γ $(= [0,\infty))$
(Theorem 14.2)

Euler's Gamma function. $\Gamma(z) := \displaystyle\int_0^\infty t^{z-1} e^{-t}\, dt$

Theorem. $\displaystyle\int_0^\infty t^{z-1} e^{-t}\, dt$ defines a holomorphic function on

$$D = \{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$$

Remarks: (1) $t^{z-1} = e^{(z-1)\ln(t)}$ is not defined at 0.

So, hypothesis #3 of Theorem 14.2 also has to hold near 0.

Meaning: here $\displaystyle\int_0^\infty t^{z-1} e^{-t}\, dt = \lim_{\substack{r \to 0 \\ R \to \infty}} \int_r^R e^{-t} t^{z-1}\, dt$

$\rightsquigarrow$ the convergence of these limits must be uniform on every compact subset A contained in D.

(2) See Problem #3 of Homework 7:

for $z = n \geq 1$ integer: $\Gamma(n) = \displaystyle\int_0^\infty t^{n-1} e^{-t}\, dt = (n-1)!$ (remember $0! = 1$)

Thus $\Gamma(z)$ extends the definition of factorial to all complex numbers with positive real part.

(15.3) Proof of Theorem (15.2). We need to verify the three hypotheses listed in Theorem (14.2). That is, for

$$K(t,z) = t^{z-1} e^{-t} = e^{(z-1)\ln(t)} e^{-t}$$: complex valued

function of $t \in \mathbb{R}$ $(t>0)$ and $z \in D$ $(= \mathrm{Re}(z) > 0)$, we have to check:

(1) For fixed t, $K(t,z)$ is a holomorphic function of z (obvious, since exponential is holomorphic)

(2) $$\left.\begin{aligned} K(t,z) &= e^{(z-1)\ln t} e^{-t} \\ \frac{\partial}{\partial z} K(t,z) &= \ln(t)\, e^{(z-1)\ln t} e^{-t} \end{aligned}\right\}$$ are continuous functions of both t and z (again it is clear)

(3) This is the difficult part of Theorem (15.2). See remark (1) on the previous page to figure out what we have to prove.

To prove: Given a closed and bounded set A contained in D and $\varepsilon > 0$, there exist $r_1 > 0$ and $R_1 > 0$ such that

$$(\star)\quad \left| \int_r^{r'} K(t,z)\,dt \right| < \varepsilon$$

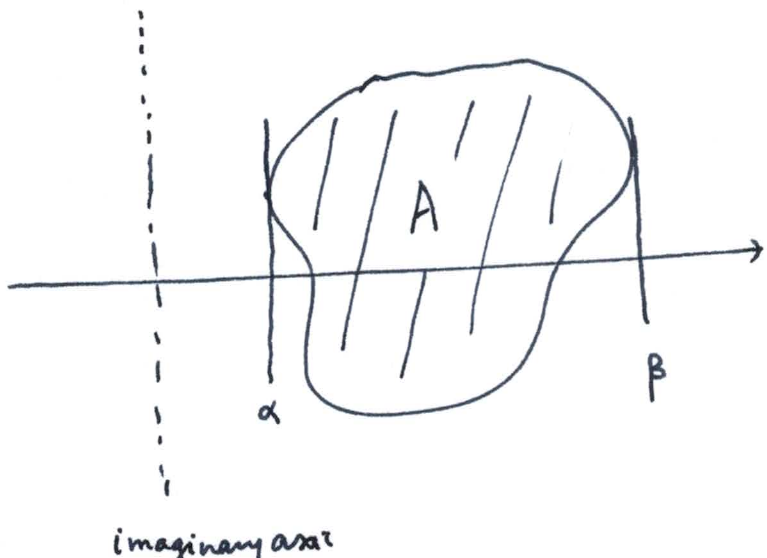
$$(\star\star)\quad \left| \int_R^{R'} K(t,z)\,dt \right| < \varepsilon$$

for every $0 < r \le r' < r_1$, $R_1 < R \le R'$, $z \in A$

(same for $\frac{\partial K}{\partial z}(t,z)$)

Proof of (☆) for $K(t,z)$:

Since A is closed and bounded, we can choose ~~$0 < L < R$ so~~ $0 < \alpha < \beta$ such that $\alpha \leqslant Re(z) \leqslant \beta$ for every $z \in A$.



For $t < 1$

$\ln(t) < 0$

$$\Rightarrow (Re(z)-1)\ln(t) \leqslant (\alpha-1)\ln(t)$$

$$|t^{z-1}| = |e^{(z-1)\ln(t)}| \leqslant t^{\alpha-1} \quad \text{for every } z \in A. \quad \text{and } e^{-t} \leqslant 1 \Rightarrow$$

Hence $(0 \lneq r \leq r' \lneq 1)$
$$\left|\int_r^{r'} t^{z-1}e^{-t}\,dt\right| \leqslant \left|\int_r^{r'} t^{\alpha-1}\,dt\right| = \frac{(r')^\alpha - r^\alpha}{\alpha}$$

As $x^\alpha \to 0$ (as $x \to 0$, for $\alpha > 0$) we can choose ~~$0 < r < r'$~~ r_1 so that $\forall\ 0 \lneq r \leq r' < r_1$ we have $\frac{(r')^\alpha - r^\alpha}{\alpha} < \varepsilon$.

(That is, pick $r_1 > 0$ so that for every $0 < r < r_1$, $r^\alpha < \frac{\varepsilon\alpha}{2}$.

then for every $0 < r \leq r' < r_1$ $\quad \frac{r'^\alpha - r^\alpha}{\alpha} < \frac{r'^\alpha + r^\alpha}{\alpha} = \frac{\varepsilon\alpha}{\alpha} = \varepsilon$)

Proof of (☆) for $\dfrac{\partial K}{\partial z} = \ln(t)\, t^{z-1} e^{-t}$:

Use the same argument to arrive at

$$\left| \int_r^{r'} \ln(t)\, t^{z-1} e^{-t}\, dt \right| \leq \left| \int_r^{r'} \ln(t)\, t^{\alpha-1}\, dt \right|$$

Then check that $\displaystyle\int t^{\alpha-1} \ln(t)\, dt = \frac{t^{\alpha}}{\alpha} \ln(t) - \frac{t^{\alpha}}{\alpha^2}$
(integration by parts)

After that follow the same argument and the fact that

$\lim\limits_{x \to 0^+} x^{\alpha} \ln(x) = 0$ (for $\alpha > 0$).

Proof of (☆☆) for $K(t,z)$. If $t > 1$ so that $\ln(t) > 0$

$$|t^{z-1}| \leq t^{\beta-1} \quad \text{for every } z \in A.$$

Now $\lim\limits_{t \to \infty} t^{\beta-1} e^{-t/2} = 0$. Let t_0 be so that $t^{\beta-1} e^{-\frac{t}{2}} \leq 1$ for every $t > t_0$

Then $\displaystyle\left| \int_R^{R'} t^{z-1} e^{-t}\, dt \right| \leq \left| \int_R^{R'} (t^{\beta-1} e^{-t/2})\, e^{-t/2}\, dt \right|$

$t_0 \leq R < R'$

$$\leq \int_R^{R'} e^{-t/2}\, dt = \frac{e^{-R'/2} - e^{-R/2}}{-1/2}$$

$$= 2\left(e^{-R/2} - e^{-R'/2}\right)$$

Again $e^{-R/2} \to 0$ as $R \to \infty$. So pick $R_1 \geq t_0$ such that $e^{-R/2} < \dfrac{\varepsilon}{4}$ for every $R > R_1$

Then $2\left(e^{-R/2} - e^{-R'/2}\right) < 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon.$

Proof of (☆☆) for $\dfrac{\partial K}{\partial z}$ is left as an exercise.

(15.4) $\Gamma(z+1) = z\,\Gamma(z)$ $\quad (\mathrm{Re}(z) > 0)$

Proof

$$\Gamma(z+1) - z\Gamma(z) = \int_0^\infty t^z e^{-t}\,dt - \int_0^\infty z t^{z-1} e^{-t}\,dt$$

Now $t^z e^{-t} - z t^{z-1} e^{-t} = -\dfrac{d}{dt}\left(t^z e^{-t}\right)$. So by fundamental theorem of Calculus

$$\Gamma(z+1) - z\Gamma(z) = \Big[-t^z e^{-t}\Big]_{\lim_{t\to 0}}^{\lim_{t\to\infty}}$$

Both these limits are zero. Hence $\Gamma(z+1) - z\Gamma(z) = 0$ $\square$

(15.5) Compute $\Gamma\left(\frac{1}{2}\right)$:

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t}\,dt$$

Set $u = t^{1/2}$, $u^2 = t$, $\quad dt = 2u\,du$

$$= \int_0^\infty \frac{1}{u} e^{-u^2}\, 2u\,du = 2\int_0^\infty e^{-u^2}\,du$$

Gaussian integral

$$= \cancel{2}\,\frac{\sqrt{\pi}}{\cancel{2}} = \sqrt{\pi}.$$