# Lecture 16



(16.0) Recall : we defined the gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$

and proved that (1) $\Gamma(n) = (n-1)!$ ($n \geq 1$ integer)

(2) $\Gamma(z)$ is a holomorphic function of $z$ in the domain

$$D = \{ z \in \mathbb{C} \mid Re(z) > 0 \}$$

(3) $\Gamma(z+1) = z\,\Gamma(z)$

In this lecture we give an alternate expression of gamma function.

(16.1) Consider the function

$$g(z) = z \cdot e^{\gamma z} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right) e^{-z/n} \right\}$$

($\gamma$ is a positive real number defined in (16.21))

Notion of infinite product : the notation $\prod_{n=1}^{\infty} u_n(z)$ stands for infinite product $u_1(z) \cdot u_2(z) \cdot \ldots$ . More precisely, it means

$$\prod_{n=1}^{\infty} u_n(z) = \lim_{m \to \infty} \prod_{n=1}^{m} u_n(z)$$

← limit of a sequence of functions (introduced in Lecture 7 page 6)

To say that $\prod_{n=1}^{\infty} u_n(z)$ converges uniformly on compact subsets in a domain $D$ is same as saying :

given a compact set $A$ contained in $D$ and $\varepsilon > 0$, there exists

$m_0 > 0$ such that for every $m \geq m_0$ and $p > 0$,

$$\left| \prod_{n=1}^{m+p} u_n(z) - \prod_{n=1}^{m} u_n(z) \right| < \varepsilon \quad \text{for every } z \in A.$$

(16.2) Euler-Mascheroni constant $\gamma$

$$\gamma = \lim_{m \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} - \ln(m+1) \right)$$

To see that this limit exists, let $x_m = \int_0^1 \frac{t}{m(m+t)}\, dt \quad (m \geq 1)$

Clearly
$$x_m = \int_0^1 \frac{m+t-m}{m(m+t)}\, dt = \int_0^1 \frac{1}{m}\, dt - \int_0^1 \frac{1}{m+t}\, dt$$
$$= \frac{1}{m} - \ln\left(\frac{m+1}{m}\right)$$

Also
$$x_m = \int_0^1 \frac{t}{m(m+t)}\, dt \leq \int_0^1 \frac{1}{m^2}\, dt = \frac{1}{m^2}$$

Since $\sum_{m=1}^{\infty} \frac{1}{m^2}$ converges, so does $\sum_{m=1}^{\infty} x_m$. Thus

$$x_1 + x_2 + \cdots + x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \big(\ln(2) + \ln(3) - \ln(2) + \cdots + \ln(n+1) - \ln(n)\big)$$
$$= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n+1)$$

So $\gamma = \sum_{m=1}^{\infty} x_m$ exists (actually $\gamma = 0.5772157\ldots$)

(16.3) Theorem. $g(z) = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$

converges uniformly on the entire complex plane (i.e. on compact subsets of $\mathbb{C}$).

Proof Let $A \subset \mathbb{C}$ be a compact set. Let $R>0$ be such that $|z| \leq R$ for every $z \in A$.

Now for $n > 2R$, we have (for every $z \in A$) (recall: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ for $|x|<1$)

$$\left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| = \left| -\frac{1}{2}\frac{z^2}{n^2} + \frac{1}{3}\frac{z^3}{n^3} - \cdots \right|$$ (since $\left|\frac{z}{n}\right| \leq \frac{1}{2}$)

$$\leq \frac{|z|^2}{n^2}\left\{1 + \left|\frac{z}{n}\right| + \left|\frac{z^2}{n^2}\right| + \cdots \right\}$$

$$\leq \frac{1}{2}\frac{R^2}{n^2}$$

Since $\sum \frac{1}{n^2}$ converges we get that $\sum_{n>2R} \log\left(1+\frac{z}{n}\right) - \frac{z}{n}$ converges uniformly (for every $z \in A$), and hence defines a holomorphic function. (see Theorem (7.3))

Take exponential to get $\prod_{n>2R} \left(1+\frac{z}{n}\right) e^{-z/n}$. Hence $g(z)$ is a holomorphic function for $z \in \mathbb{C}$. □

(16.3) Definition (Weierstrass 1856)

$$\Gamma_1(z) = \frac{1}{g(z)}.$$ We will see later that

$\Gamma_1(z) = \Gamma(z)$ for $Re(z) > 0$. Until then, we need to use notation different from $\Gamma$, therefore we will call $\frac{1}{g(z)}$. $\Gamma_1(z)$.

Some easy remarks. since $g(z)$ is holomorphic everywhere and

$$g(z) = 0 \iff z = 0, -1, -2, -3, \dots$$

(zeroes of order 1)

$\Gamma_1(z)$ is a meromorphic function on $\mathbb{C}$ with simple poles at $0, -1, -2, -3, \dots$

(16.4) $\Gamma_1(1) = 1$ and $\Gamma_1'(1) = -\gamma$

Proof.

$$g(1) = e^{\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-\frac{1}{n}}$$

$$= e^{\gamma} \lim_{m \to \infty} (1+1) e^{-1} \left(1 + \frac{1}{2}\right) e^{-\frac{1}{2}} \dots \left(1 + \frac{1}{m}\right) e^{-\frac{1}{m}}$$

$$= e^{\gamma} \lim_{m \to \infty} \left\{ \left(2 \cdot \frac{3}{2} \cdot \frac{4}{3} \dots \frac{m+1}{m}\right) e^{-1-\frac{1}{2} \dots - \frac{1}{m} + \ln(m+1)} \cdot e^{-\ln(m+1)} \right\}$$

$$= e^{\gamma} \lim_{m \to \infty} e^{-\left(1 + \frac{1}{2} + \dots + \frac{1}{m} - \ln(m+1)\right)} (m+1) \cdot e^{-\ln(m+1)}$$

$$= e^{\gamma} e^{-\gamma} = 1.$$

Now $\Gamma_1(z) = \dfrac{1}{g(z)} \Rightarrow \Gamma_1'(z) = \dfrac{-g'(z)}{g(z)^2}$

$\dfrac{\Gamma_1'(z)}{\Gamma_1(z)} = \dfrac{-g'(z)}{g(z)}$. For a holomorphic (or meromorphic) function $f(z)$, $\dfrac{f'(z)}{f(z)}$ is called logarithmic derivative. Check that (by Leibniz rule) if $f = f_1 f_2 \cdots f_n$ then $\dfrac{f'}{f} = \dfrac{f_1'}{f_1} + \dfrac{f_2'}{f_2} + \dots + \dfrac{f_n'}{f_n}$

Thus uniform continuity ensures that this formula is valid for infinite products.

$$\Rightarrow \frac{\Gamma_1'(z)}{\Gamma_1(z)} = -\frac{g'(z)}{g(z)} = -\left[\frac{1}{z} + \gamma + \sum_{n=1}^{\infty} \frac{1}{z+n} - \frac{1}{n}\right]$$

Set $z=1$ $(\Gamma_1(1)=1)$ : $\Gamma_1'(1) = -\gamma - \left[1 + \sum_{n=1}^{\infty}\left(\frac{1}{1+n} - \frac{1}{n}\right)\right]$

$$= -\gamma - \lim_{m\to\infty}\left(1 + \frac{1}{2} - 1 + \frac{1}{3} - \frac{1}{2} + \dots + \frac{1}{m+1} - \frac{1}{m}\right)$$

$$= -\gamma - \lim_{m\to\infty} \frac{1}{m+1} = -\gamma$$

$\square$

(16.5) Difference equation satisfied by $\Gamma_1(z)$.

$$\Gamma_1(z+1) = z\,\Gamma_1(z)$$

Proof $\quad z\,g(z+1) = z(z+1)\,e^{\gamma(z+1)} \prod_{n=1}^{\infty}\left(1+\frac{z+1}{n}\right)e^{-(z+1)/n}$

$$= e^{\gamma z}\cdot z \cdot \lim_{m\to\infty} e^{\gamma}\,(z+1)(z+2)\frac{(z+3)}{2}\cdots\frac{z+m+1}{m}\cdot e^{\left(-1-\frac{1}{2}\cdots-\frac{1}{m}\right)(z+1)}$$

$$= z\,e^{\gamma z}\,e^{\gamma}\lim_{m\to\infty}\left[\left\{\prod_{n=1}^{m+1}\left(1+\frac{z}{n}\right)e^{-z/n}\right\}.(m+1)\,e^{-1-\frac{1}{2}\cdots-\frac{1}{m}}\,e^{\frac{z}{m+1}}\right]$$

$m+1 = e^{\ln(m+1)}$. So in the limit we get.

$$z\,e^{\gamma z}\,e^{\gamma}\left\{\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right\}\,e^{-\gamma}\lim_{m\to\infty} e^{z/m+1} = g(z)$$

So $\quad \Gamma_1(z+1) = \dfrac{1}{g(z+1)}\cdot\dfrac{z}{z} = \dfrac{z}{g(z)} = z\,\Gamma_1(z) \qquad \square$

(16.6) Relation with trigonometric functions

$$\Gamma_1(z)\,\Gamma_1(1-z) = \frac{\pi}{\sin(\pi z)}$$

In order to prove this we need to work out an expression of $\dfrac{\sin(z)}{z}$ as an infinite product.

# (16.7) A generalization of partial fractions 

Let $f(z)$ be a meromorphic function which only has <u>simple poles</u> $a_1, a_2, a_3, \ldots$ arranged so that $|a_1| \leq |a_2| \leq \ldots$ and assume $0$ is not a pole. (Remember: simple pole = pole of order 1).

Let $b_n = \operatorname{Res}_{z=a_n} f(z) \qquad (n = 1, 2, 3, \ldots)$

Further assume there are positive real numbers $R_1 < R_2 < R_3 \cdots$ $\lim_{n\to\infty} R_n = \infty$ such that

(1) None of the poles lie on $C_{R_n}$ = circle of radius $R_n$ centered at 0. $(n = 1, 2, 3, \ldots)$

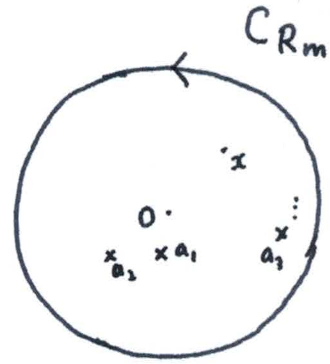
(2) $|f(z)|$ is bounded on $C_{R_n}$ $(n=1,2,3,\ldots)$. That is, there is a number $M > 0$ such that $|f(z)| \leq M \quad \forall z \in C_{R_n}$ $(n=1,2,3,\ldots)$

Then $$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left( \frac{1}{z-a_n} + \frac{1}{a_n} \right)$$

Proof. Let $x \in \mathbb{C}$, $x \neq a_n$ for any $n$. Choose $m_0$ large enough so that $x$ is within the circle $C_{R_{m_0}}$ (and hence also within $C_{R_m}$ with $m \geq m_0$)

Then for any $m \geq m_0$, we will have

$$f(x) + \sum_{\substack{k \text{ such} \\ \text{that } a_k \text{ is} \\ \text{within } C_{R_m}}} \frac{b_k}{a_k - x} \underset{\text{Cauchy's formula}}{=} \frac{1}{2\pi i} \int_{C_{R_m}} \frac{f(z)}{z-x}\, dz$$




Write $\dfrac{1}{z-x} = \dfrac{1}{z} + \dfrac{x}{z(z-x)}$ to get

$$\frac{1}{2\pi i} \int_{C_{R_m}} \frac{f(z)}{z-x}\, dz = \frac{1}{2\pi i} \int_{C_{R_m}} \frac{f(z)}{z}\, dz + \frac{1}{2\pi i} \cdot x \cdot \int_{C_{R_m}} \frac{f(z)}{z(z-x)}\, dz$$

$$= f(0) + \frac{x}{2\pi i} \int_{C_{R_m}} \frac{f(z)}{z(z-x)}\, dz + \sum_{\substack{k \text{ such} \\ \text{that } a_k \\ \text{is within} \\ C_{R_m}}} \frac{b_k}{a_k}$$

Hence we get

$$f(x) - f(0) + \sum_{\substack{k \text{ as} \\ \text{before}}} b_k \left\{ \frac{1}{a_k - x} - \frac{1}{a_k} \right\} = \frac{x}{2\pi i} \int_{C_{R_m}} \frac{f(z)}{z(z-x)}\, dz$$

But $$\left| \int_{C_{R_m}} \frac{f(z)}{z(z-x)}\, dz \right| \leq \frac{M \cdot 2\pi R_m}{R_m (R_m - |x|)} \to 0 \text{ as } m \to \infty$$

$$\Rightarrow \quad f(x) = f(0) + \sum_{\substack{\text{all } k \\ k = 1, 2, \dots}} b_k \left( \frac{1}{x - a_k} + \frac{1}{a_k} \right) \qquad \square$$

e.g. $f(z) = \operatorname{cosec}(z) - \frac{1}{z}$

Poles : $\{z = n\pi,\ n \in \mathbb{Z},\ n \neq 0\}$ $\quad$ Residue at $z = n\pi$ $= (-1)^n$

Let $R_m = \left(m + \frac{1}{2}\right)\pi$. Check: $|f(z)|$ is bounded on circles $C_{R_m}$.

Hence $$f(z) = f(0) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^n \left( \frac{1}{z - n\pi} + \frac{1}{n\pi} \right)$$

But $$f(0) = \lim_{z \to 0} \frac{1}{\sin z} - \frac{1}{z} = \lim_{z \to 0} \frac{z - \sin(z)}{z \sin(z)} = 0.$$

$$\Rightarrow \quad \operatorname{Cosec}(z) = \frac{1}{z} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^n \left( \frac{1}{z - n\pi} + \frac{1}{n\pi} \right)$$

(16.8) A general theorem about infinite products

Let $F(z)$ be a holomorphic function on the entire complex plane.

Assume $F$ only has zeroes of multiplicity 1 at $a_1, a_2, a_3, \ldots$

$0 \neq a_n$ (for any $n$) and $\lim_{n \to \infty} |a_n| = \infty$.

Let $f(z) = \frac{F'(z)}{F(z)}$. Then $f$ only has simple poles at $a_n$'s with residue = multiplicity of zero at $a_n$'s $= 1$.

Assuming the existence of $R_m$'s as in section (16.7) for $f(z)$, we get

$$f(z) = f(0) + \sum_{n=1}^{\infty} \left( \frac{1}{z-a_n} + \frac{1}{a_n} \right) . \quad \text{Now } f = \frac{F'}{F}$$

$$\frac{F'(z)}{F(z)} = \frac{F'(0)}{F(0)} + \sum_{n=1}^{\infty} \frac{1}{z-a_n} + \frac{1}{a_n} \quad \leftarrow \text{logarithmic derivative}$$

$$\Rightarrow \quad F(z) = c\, e^{\frac{F'(0)}{F(0)} \cdot z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}} \quad \left(\text{integrate and exponentiate}\right)$$

setting $z = 0$ gives $c = F(0)$.

e.g. $F(z) = \dfrac{\sin(z)}{z}$ zeroes at $z = n\pi$ $\left(\begin{matrix} n \in \mathbb{Z} \\ n \neq 0 \end{matrix}\right)$ (of mult. 1)

$$F(0) = 1 \qquad F'(z) = \frac{z\cos(z) - \sin(z)}{z^2} \xrightarrow[z \to 0]{\lim} 0$$

$$\frac{F'(z)}{F(z)} = \cot(z) - \frac{1}{z} \qquad \left(\text{again take } R_m = \left(m + \tfrac{1}{2}\right)\pi\right)$$

So
$$\frac{\sin(z)}{z} = \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}}$$

$$= \left\{ \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi}\right) e^{z/n\pi} \right\} \left\{ \prod_{n=1}^{\infty} \left(1 + \frac{z}{n\pi}\right) e^{-\frac{z}{n\pi}} \right\}$$

(16.9)

Proof of (16.6) formula $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$ 

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n} \cdot \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

$$= \frac{-1}{z^2}\left[(-z)\,e^{-\gamma z}\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}\right]\left[z\,e^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right]$$

$$= -\frac{g(-z)\cdot g(z)}{z^2}$$

$$\Rightarrow \quad -z\,\frac{\sin(\pi z)}{\pi} = \frac{1}{\Gamma(-z)\,\Gamma(z)}$$

$$\Rightarrow \quad \left(-z\,\Gamma(-z)\right)\cdot\Gamma(z) = \frac{\pi}{\sin(\pi z)}$$

$$\Rightarrow \quad \boxed{\Gamma(1-z)\,\Gamma(z) = \frac{\pi}{\sin(\pi z)}}$$