Lecture 17

(17.0) Recall that last time we defined

$$\Gamma_1(z) = \left[z\, e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n} \right]^{-1}$$

where $\gamma = \lim_{m\to\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{m} - \ln(m+1)\right)$ is Euler-Mascheroni constant.

- Γ_1 is a meromorphic function with simple poles at $0, -1, -2, \dots$
- $\Gamma_1(n) = (n-1)!$ for $n \geq 1$ integer
- $\Gamma_1(z+1) = z\,\Gamma_1(z)$
- $\Gamma_1(z)\,\Gamma_1(1-z) = \dfrac{\pi}{\sin(\pi z)}$

(17.1) $$\frac{1}{\Gamma_1(z)} = z \left[\lim_{m\to\infty} e^{\left(1+\frac{1}{2}+\dots+\frac{1}{m} - \ln(m+1)\right)\cdot z} \right] \cdot \left[\lim_{m\to\infty} \prod_{n=1}^{m} \left(1+\frac{z}{n}\right) e^{-z/n} \right]$$

$$= z \cdot \lim_{m\to\infty} \left[e^{\left(1+\frac{1}{2}+\dots+\frac{1}{m} - \ln(m+1)\right) z} \cdot \prod_{n=1}^{m} \left(1+\frac{z}{n}\right) e^{-z/n} \right]$$

$$= z \cdot \lim_{m\to\infty} \left[(m+1)^{-z} \prod_{n=1}^{m} \left(1+\frac{z}{n}\right) \right]$$

$$= z \cdot \lim_{m \to \infty} \left[\prod_{n=1}^{m} \left(1+\frac{1}{n}\right)^{-z} \cdot \prod_{n=1}^{m} \left(1+\frac{z}{n}\right) \right]$$

$$\Rightarrow \boxed{\Gamma_1(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left\{ \left(1+\frac{z}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^{z} \right\}}$$

(17.2) $\Gamma(z) = \Gamma_1(z)$ for $Re(z) > 0$. Recall that we defined in Lecture 15: $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (Re(z) > 0)$

To see this, let us define

$$G(z,n) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1} dt \qquad (Re(z) > 0)$$

Set $t = n \cdot u$ to get $G(z,n) = \int_0^1 (1-u)^n (nu)^{z-1} \cdot n \, du$

$$= n^z \int_0^1 (1-u)^n u^{z-1} du$$

Now $\int_0^1 (1-u)^n u^{z-1} du$ can be evaluated by using integration by parts.

$$\int (1-u)^n u^{z-1}\,du = (1-u)^n \frac{u^z}{z} + \frac{n}{z}\int (1-u)^{n-1} u^z\,du$$

Substituting limits $u = 0, 1$ ($u^z = e^{z\ln(u)} \to 0$ as $u \to 0$ since $Re(z) > 0$)

we get

$$\int_0^1 (1-u)^n u^{z-1}\,du = \frac{n}{z}\int_0^1 (1-u)^{n-1} u^z\,du = \dots = \frac{n(n-1)\dots 1}{z(z+1)\dots(z+n-1)}\int_0^1 u^{z+n-1}\,du$$

$$\Rightarrow \quad G(z,n) = \frac{n!}{z(z+1)\dots(z+n)} \cdot n^z$$

By Homework 8, problem , we see that

$$\lim_{n\to\infty} G(z,n) = \Gamma_1(z) \quad , \text{ i.e. for } Re(z) > 0$$

$$\Gamma_1(z) = \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt$$

$$\Rightarrow \quad \Gamma(z) - \Gamma_1(z) = \lim_{n\to\infty}\left[\int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) . t^{z-1}\,dt + \int_n^\infty e^{-t} t^{z-1}\,dt\right]$$

Now $\lim_{n\to\infty} \int_n^\infty e^{-t} t^{z-1}\,dt = 0$ by virtue of the fact that

$$\int_0^\infty e^{-t}\, t^{z-1}\, dt$$ exists. Thus it remains to prove that

$$\lim_{n\to\infty} \int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n \right) \cdot t^{z-1}\, dt = 0.$$

Important inequality $0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n}$. This inequality implies that

$$\left| \int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n \right) \cdot t^{z-1}\, dt \right| \leq \frac{1}{n} \int_0^n t^{x+1} e^{-t}\, dt$$

$x = \mathrm{Re}(z) > 0$

$$< \frac{1}{n} \underbrace{\int_0^\infty t^{x+1} e^{-t}\, dt}_{\text{exists as a finite number}} \to 0 \quad \text{as } n \to \infty$$

Proof of Important inequality: Let $0 \leq y < 1$. Then

$$1 + y \leq e^{y} \leq (1-y)^{-1} \quad - \quad (\star)$$

Since $e^{y} = 1 + y + \left(\frac{y^2}{2!} + \dots + \frac{y^k}{k!} + \dots \right) \geq 1 + y$ for $y \geq 0$

$$(1-y)^{-1} = 1 + y + y^2 + \dots \geq 1 + y + \frac{y^2}{2!} + \dots = e^{y}$$

(for $0 \leq y < 1$)

Set $y = \frac{t}{n}$ for $0 \leq t < n$ in (★) to get

$$1 + \frac{t}{n} \leq e^{t/n} \leq \left(1 - \frac{t}{n}\right)^{-1} \Rightarrow \left(1 + \frac{t}{n}\right)^{n} \leq e^{t} \leq \left(1 - \frac{t}{n}\right)^{-n}$$

$$\Rightarrow \quad \left(1 + \frac{t}{n}\right)^{-n} \geq e^{-t} \geq \left(1 - \frac{t}{n}\right)^{n}$$

So, $\quad 0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^{n} = e^{-t}\left(1 - e^{t}\left(1 - \frac{t}{n}\right)^{n}\right)$

$$\leq e^{-t}\left(1 - \left(1 + \frac{t}{n}\right)^{n}\left(1 - \frac{t}{n}\right)^{n}\right) \quad \text{since } e^{t} \geq \left(1 + \frac{t}{n}\right)^{n}$$

$$= e^{-t}\left(1 - \left(1 - \frac{t^2}{n^2}\right)^{n}\right)$$

Now $(1-a)^n \geq 1 - na$ for $a \geq 0$. (Check this: by induction on n)

So $\quad 0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^{n} \leq e^{-t}\, n\, \frac{t^2}{n^2}$ as required

□

(17.3) From now onwards we drop the subscript 1 from $\Gamma_1(z)$.

$$\Gamma_1(z) = \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt \quad \text{for } \operatorname{Re}(z) > 0$$

$$= \left[z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} \right]^{-1} \quad \text{as a mero. function of } z \in \mathbb{C}.$$