Lecture 18

(18.0) Recall that we defined $\Gamma(z) = \dfrac{1}{z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}}$

- $\Gamma(z+1) = z\,\Gamma(z)$
- $\Gamma(n) = (n-1)!$ for $n \geq 1$ integer
- $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$

Finally $\Gamma(z) = \displaystyle\int_0^{\infty} t^{z-1} e^{-t}\, dt$ for $\mathrm{Re}(z) > 0$

(18.1) Application to Euler's integral

Consider the problem of computing $\displaystyle\int_0^{\pi/2} \cos^m\theta \sin^n\theta\, d\theta$

($m, n \in \mathbb{R}$ positive real). Upon performing change of variables $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$, we get

$$I_{m,n} = \int_0^{\pi/2} \cos^m\theta \sin^n\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \cos^{m-1}(\theta) \sin^{n-1}(\theta) \left(2\sin\theta\cos\theta\, d\theta\right)$$

$$= \frac{1}{2}\int_0^1 (1-x)^{\frac{m-1}{2}} x^{\frac{n-1}{2}}\, dx$$

if $m = 2p-1$, $n = 2q-1$ we get

$$\int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta\, d\theta = \frac{1}{2}\int_0^1 (1-x)^{p-1} x^{q-1}\, dx \quad - (\star)$$

This integral was studied by Euler around 1770's

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \quad (\text{for real positive } p \text{ and } q)$$

(as always $x^{p-1} = e^{(p-1)\ln(x)}$) [Note: set $y = 1-x$ to see that $B(p,q) = B(q,p)$]

(18.2) Theorem $$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

Proof. $$\Gamma(p)\,\Gamma(q) = \int_0^\infty e^{-t}\,t^{p-1}\,dt \int_0^\infty e^{-t}\,t^{q-1}\,dt$$

If we set $t = x^2$, $\int_0^\infty e^{-t}\,t^{a-1}\,dt$ becomes $2\cdot\int_0^\infty e^{-x^2}\,x^{2a-1}\,dx$

So $$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty e^{-x^2}\,x^{2p-1}\,dx \int_0^\infty e^{-y^2}\,y^{2q-1}\,dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-x^2-y^2}\,x^{2p-1}\,y^{2q-1}\,dx\,dy$$ ← integral over the first quadrant in x-y plane

Change to polar coordinates:

$x = r\cos\theta$ $\qquad dx\,dy \rightsquigarrow r\,dr\,d\theta$

$y = r\sin\theta$

$(0 \leq \theta \leq \frac{\pi}{2},\ r \geq 0)$

$$\Gamma(p)\Gamma(q) = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} (r\cos\theta)^{2p-1} (r\sin\theta)^{2q-1}\, r\, dr\, d\theta$$

$$= 4\int_0^{\infty} e^{-r^2}\, r^{2(p+q)-1}\, dr \cdot \int_0^{\pi/2} \cos^{2p-1}\theta\, \sin^{2q-1}\theta\, d\theta$$

Now $2\int_0^{\infty} e^{-r^2}\cdot r^{2(p+q)-1}\, dr = \Gamma(p+q)$ by second line of the proof on previous page.

and $2\int_0^{\pi/2} \cos^{2p-1}\theta\, \sin^{2q-1}\theta\, d\theta = B(q,p) = B(p,q)$ by (☆) on page 1.

So $\Gamma(p)\,\Gamma(q) = B(p,q)\cdot\Gamma(p+q)$ as claimed. $\square$

(18.2) Gauss' formula for $\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$:

$$\psi(z) = \int_0^{\infty}\left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt \qquad (\text{for } \operatorname{Re}(z)>0)$$

Proof of Gauss' formula. Recall from lecture 16, page 5 :

$$\psi(z) = -\gamma - \frac{1}{z} + \lim_{m\to\infty}\left(\sum_{n=1}^{m} \frac{1}{n} - \frac{1}{z+n}\right)$$

Now $\dfrac{1}{z+n} = \displaystyle\int_0^{\infty} e^{-t(z+n)}\, dt$

So $\psi(z) = -\gamma - \int_0^\infty e^{-zt}\,dt + \lim_{m\to\infty}\left[\sum_{n=1}^{m} \int_0^\infty (e^{-nt} - e^{-(z+n)t})\,dt\right]$

Now $x + x^2 + \dots + x^m = x\left(\dfrac{x^m - 1}{x-1}\right) = \dfrac{x(1-x^m)}{1-x}$ $\quad x = e^{-t}$ here

$$\psi(z) = -\gamma + \lim_{m\to\infty} \int_0^\infty \frac{e^{-t} - e^{-zt} - e^{-(m+1)t} + e^{-(z+m+1)t}}{1-e^{-t}}\,dt$$

A formula for γ : $\boxed{\gamma = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) e^{-t}\,dt}$

$$\Rightarrow \psi(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt - \lim_{m\to\infty} \int_0^\infty e^{-(m+1)t}\,\frac{(1-e^{-zt})}{1-e^{-t}}\,dt$$

Now $\dfrac{1-e^{-zt}}{1-e^{-t}} \to z$ as $t \to 0$ so it is a continuous function on $[0,\infty)$.

It is clearly bounded, since for $t \ge 1$, $\left|\dfrac{1-e^{-zt}}{1-e^{-t}}\right| \le \dfrac{1+|e^{-zt}|}{1-e^{-1}} < \dfrac{2}{1-e^{-1}}$ $\quad (\mathrm{Re}(z) > 0)$

$$\left|\int_0^\infty e^{-(m+1)t}\left(\frac{1-e^{-zt}}{1-e^{-t}}\right) dt\right| \le K \cdot \int_0^\infty e^{-(m+1)t}\,dt = \frac{K}{m+1} \to 0 \text{ as } m \to \infty.$$

Some constant

$$\Rightarrow \psi(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt$$

□

(18.3) Binet's formula for $\log \Gamma(z)$ $\leftarrow$ application of Gauss' formula for $\psi(z)$. Again for $\mathrm{Re}(z) > 0$.

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi) + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-tz}}{t}\,dt$$

Note: $\log \Gamma(z)$ is defined as $-\left[\log z + \gamma z + \sum_{n=1}^{\infty}\left(\log\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right)\right]$

These are $\log(w) = \ln(|w|) + i\arg(w)$
$-\pi < \arg(w) < \pi$

To derive Binet's formula, we need the following:

$$\log(z) = \int_0^\infty \frac{e^{-t} - e^{-zt}}{t}\,dt$$

Let us start from Gauss' expression for $\psi(z+1)$

$$\psi(z+1) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}e^{-t}}{1 - e^{-t}}\right)dt = \int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-zt}}{e^t - 1}\,dt$$

$$= \int_0^\infty \left\{\frac{e^{-t} - e^{-zt}}{t} + \frac{1}{2}e^{-zt} - \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)e^{-zt}\right\}dt$$

$$= \log(z) + \frac{1}{2z} - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)e^{-zt}\,dt$$

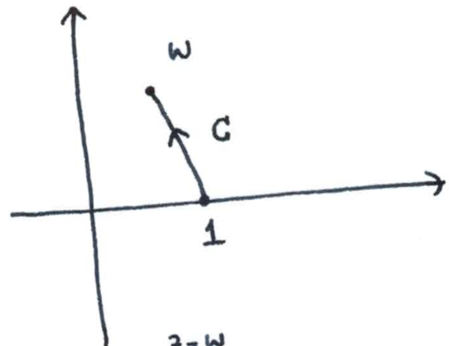
$\psi(z+1) = \psi(z) + \frac{1}{z}$ from Homework 8. So we get

$$\psi(z) = \log(z) - \frac{1}{2z} - \int_0^\infty \underbrace{\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)} e^{-zt}\, dt$$

continuous and bounded on $[0,\infty)$

Last integral converges uniformly for $\mathrm{Re}(z) > 0$. To get $\log\Gamma(z)$ from last equation, we integrate it from 1 to w: ($\mathrm{Re}(w) > 0$).

$$\log\Gamma(w) = \int_C \log(z)\,dz - \frac{1}{2}\int_C \frac{1}{z}\,dz - \int_C \int_0^\infty f(t)\, e^{-zt}\, dt$$



$$f(t) = \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}$$

$$= \Big[z\log(z) - z\Big]_{z=1}^{z=w} - \frac{1}{2}\Big[\log(z)\Big]_{z=1}^{z=w} - \int_0^\infty f(t)\left[\frac{e^{-zt}}{-t}\right]_{z=1}^{z=w} dt$$

flipping the order of integration is allowed by uniform convergence.

$$= w\log(w) - w + 1 - \frac{1}{2}\log(w)$$

$$+ \int_0^\infty f(t)\, \frac{e^{-wt} - e^{-t}}{t}\, dt$$

Easy check: $f(0) = 0$ $\left(\text{i.e. } \lim_{t\to 0}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) = 0\right)$

$\lim_{t\to\infty} f(t) = \frac{1}{2}$. So $\frac{f(t)}{t}$ is still continuous and bounded function on $[0,\infty)$

One last formula: $$\int_0^\infty \frac{f(t)}{t} e^{-t}\,dt = 1 - \frac{1}{2}\log(2\pi)$$

$\Rightarrow$ Binet's expression:

$$\log\Gamma(w) = \left(w - \frac{1}{2}\right)\log(w) - w + \frac{1}{2}\log(2\pi) + \int_0^\infty \frac{f(t)}{t} e^{-tw}\,dt$$

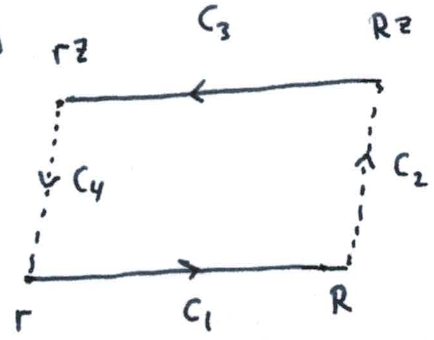
(18.4) Proof of $\log(z) = \displaystyle\int_0^\infty \frac{e^{-t} - e^{-tz}}{t}\,dt$ for $\mathrm{Re}(z) > 0$:

$$\int_0^\infty \frac{e^{-t} - e^{-tz}}{t}\,dt = \lim_{\substack{r\to 0\\ R\to\infty}} \left[\int_r^R \frac{e^{-t}}{t}\,dt - \int_r^R \frac{e^{-tz}}{t}\,dt\right]$$

set $u = tz$

$$= \lim_{\substack{r\to 0\\ R\to\infty}} \left[\int_r^R \frac{e^{-t}}{t}\,dt - \int_{rz}^{Rz} \frac{e^{-u}}{u}\,du\right]$$

$$= \lim_{\substack{r\to 0\\ R\to\infty}} \left[\int_r^{rz} \frac{e^{-t}}{t}\,dt - \int_R^{Rz} \frac{e^{-t}}{t}\,dt\right]$$ because



$$\int_{C_1+C_2+C_3+C_4} \frac{e^{-t}}{t}\,dt = 0$$ by Cauchy's formula

$$= \lim_{r\to 0} \int_r^{rz} \frac{e^{-t}}{t}\,dt \qquad \left(\text{since } \int_R^{Rz} \frac{e^{-t}}{t}\,dt \to 0 \text{ as } R\to\infty\right)$$

$$= \lim_{r\to 0} \int_r^{rz} \left(\frac{e^{-t}-1}{t} + \frac{1}{t}\right) dt$$

$$= \log(z) + \cancel{\lim_{r\to 0} \int_r^{rz} \frac{e^{-t}-1}{t}\,dt} \to 0$$

defined at 0

(18.5) Proof of $\displaystyle\int_0^\infty \frac{f(t)}{t} e^{-t}\, dt = 1 - \frac{1}{2}\log(2\pi)$.

Before using this formula, we had (on page 6)

$$\log \Gamma(w) = w\log(w) - w + 1 - \frac{1}{2}\log(w) + \int_0^\infty \frac{f(t)}{t} e^{-wt}\, dt - \int_0^\infty \frac{f(t)}{t} e^{-t}\, dt$$

Set $w = \frac{1}{2}$ and let $I = \displaystyle\int_0^\infty \frac{f(t)}{t} e^{-t}\, dt \quad J = \int_0^\infty \frac{f(t)}{t} e^{-\frac{1}{2}t}\, dt$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$:

$$\frac{1}{2}\log(\pi) = \frac{1}{2} + J - I \quad \text{— } 1$$

Write $I = \displaystyle\int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-t}}{t}\, dt = \int_0^\infty \left(\frac{1}{2} - \frac{2}{u} + \frac{1}{e^{u/2}-1}\right) e^{-u/2}\frac{du}{u} \quad (u = 2t)$

$$\Rightarrow J - I = \int_0^\infty \left(\frac{1}{t} + \frac{1}{e^t - 1} - \frac{1}{e^{t/2}-1}\right) e^{-t/2}\frac{dt}{t}$$

$$= \int_0^\infty \left(\frac{1}{t} - \frac{e^{t/2}}{e^t - 1}\right)\frac{e^{-t/2}}{t}\, dt = \int_0^\infty \left(\frac{e^{-t/2}}{t} - \frac{1}{e^t - 1}\right)\frac{dt}{t}$$

$$\Rightarrow J = \int_0^\infty \left[\frac{e^{-t/2}}{t} - \frac{1}{e^t - 1} + \frac{1}{2}e^{-t} - \frac{1}{t}e^{-t} + \frac{e^{-t}}{e^t - 1}\right] dt$$

$$= \int_0^\infty \left[\frac{e^{-t/2} - e^{-t}}{t} - \frac{1}{2}e^{-t}\right]\frac{dt}{t}$$

$$= \int_0^\infty \left[-\frac{d}{dt}\left(\frac{e^{-t/2} - e^{-t}}{t}\right) + \frac{1}{2}\left(\frac{e^{-t} - e^{-t/2}}{t}\right)\right] dt$$

$$J = -\left[\frac{e^{-t/2} - e^{-t}}{t}\right]_0^\infty + \frac{1}{2}\int_0^\infty \frac{e^{-t} - e^{-\frac{1}{2}t}}{t}\,dt = \frac{1}{2} + \frac{1}{2}\log\left(\frac{1}{2}\right) \quad -(2)$$

Combining (1) and (2) gives $I = 1 - \frac{1}{2}\log(2\pi)$

(18.6) Proof of $\gamma = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) e^{-t}\,dt$ is in following steps

(left as an exercise)

Step 1. $\frac{1-(1-t)^n}{t} = \frac{1-(1-t)^n}{1-(1-t)} = 1 + (1-t) + \dots + (1-t)^{n-1}$

$$\Rightarrow \int_0^1 \left(1-(1-t)^n\right)\frac{dt}{t} = 1 + \frac{1}{2} + \dots + \frac{1}{n}. = \int_0^n \left(1-\left(1-\frac{y}{n}\right)^n\right)\frac{dy}{y} \quad (y = nt)$$

Step 2. $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln(n)\right) = \lim_{n\to\infty}\left[\int_0^n \left(1-\left(1-\frac{t}{n}\right)^n\right)\frac{dt}{t} - \int_1^n \frac{dt}{t}\right]$

$$= \lim_{n\to\infty}\left[\int_0^1 \left(1-\left(1-\frac{t}{n}\right)^n\right)\frac{dt}{t} - \int_1^n \left(1-\frac{t}{n}\right)^n \frac{dt}{t}\right]$$

$$= \int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_1^\infty \frac{e^{-t}}{t}\,dt \quad \left[\text{using the inequality } 0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n \le \frac{t^2 e^{-t}}{n}\right]$$

from Lecture 17

Step 3. $\int_0^1 \frac{1-e^{-t}}{t}\,dt = \lim_{\varepsilon\to 0}\left[\int_\varepsilon^1 \frac{1}{t}\,dt - \int_\varepsilon^\infty \frac{e^{-t}}{t}\,dt\right]$

$$- \int_1^\infty \frac{e^{-t}}{t}\,dt$$

$$= \lim_{\varepsilon\to 0}\left[\int_{1-e^{-\varepsilon}}^1 \frac{1}{t}\,dt + \int_\varepsilon^{1-e^{-\varepsilon}} \frac{dt}{t} - \int_\varepsilon^\infty \frac{e^{-t}}{t}\,dt\right]$$

$$\int_{\varepsilon}^{1-e^{-\varepsilon}} \frac{dt}{t} = \ln\left(\frac{1-e^{-\varepsilon}}{\varepsilon}\right) \to 0 \quad \text{as } \varepsilon \to 0.$$

So
$$\gamma = \lim_{\varepsilon\to 0}\left[\int_{1-e^{-\varepsilon}}^{1} \frac{dt}{t} - \int_{\varepsilon}^{\infty} \frac{e^{-t}}{t}\,dt\right]$$

set $t = 1-e^{-u}$ $\quad dt = e^{-u}\,du$

$$= \lim_{\varepsilon\to 0}\left[\int_{\varepsilon}^{\infty} \frac{e^{-u}\,du}{1-e^{-u}} - \int_{\varepsilon}^{\infty}\frac{e^{-t}}{t}\,dt\right] = \int_0^{\infty}\left[\frac{e^{-t}}{1-e^{-t}} - \frac{e^{-t}}{t}\right]dt$$

□

(18.7) Binet's expression:

$$\log\Gamma(z) = \left(z-\tfrac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi) + \int_0^{\infty} \frac{f(t)}{t}\, e^{-tz}\,dt$$

Laplace transform of

$$\frac{f(t)}{t} = \frac{1}{t}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)$$

if we write the Taylor series:

$$\frac{t}{e^t-1} = 1 - \frac{t}{2} + \sum_{n=2}^{\infty} \frac{b_n t^n}{n!}$$

Then
$$\frac{f(t)}{t} = \sum_{n=2}^{\infty} \frac{b_n}{n!} t^{n-2} \quad \rightsquigarrow \quad \int_0^{\infty} \frac{f(t)}{t} e^{-tw}\,dt =$$

by Homework 7 problem 3

$$\sum_{n=2}^{\infty} \frac{b_n}{n(n-1)} \frac{1}{w^{n-1}}$$

$$\log\Gamma(z) = \left(z-\tfrac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi) + \left(\frac{b_2}{2}\frac{1}{z} + \frac{b_3}{6}\frac{1}{z^2} + \frac{b_4}{12}\frac{1}{z^3} + \cdots\right)$$