Lecture 19

Doubly-periodic functions

(19.0) A few of the most important entire functions we encountered so far have some sort of periodicity property. e.g. $f(z) = e^z$ satisfies

$$f(z + 2\pi i) = f(z)$$

Now we begin the study of doubly-periodic functions.

Let $\tau \in \mathbb{C}$ be such that Imaginary part of τ, $\mathrm{Im}(\tau) > 0$.

$$\Lambda_\tau = \{ m + n\tau \text{ such that } m, n \in \mathbb{Z} \} \subset \mathbb{C}.$$

Definition. A doubly-periodic (or elliptic function) relative to Λ_τ is a meromorphic function $f(z)$ such that

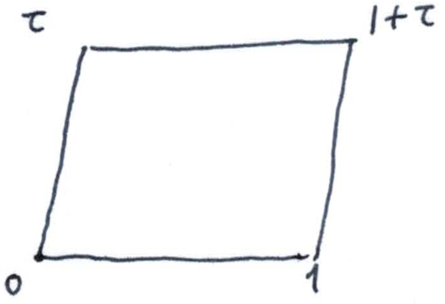
$$f(z + \ell) = f(z) \quad \text{for every } \ell \in \Lambda_\tau.$$

(That is, $f(z+1) = f(z)$, $f(z+\tau) = f(z)$ hence the name doubly-periodic)

Remark. The name 'elliptic' arose because of the significance of these functions in computing arc length of an ellipse.

(19.1) Some terminology:

fundamental parallelogram =



$z_1 = z_2$ modulo Λ_τ means $z_1 - z_2 = m + n\tau$

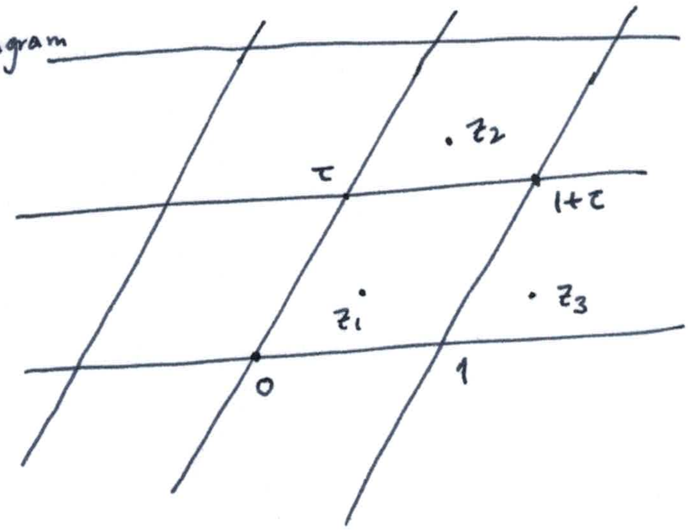
for some $m, n \in \mathbb{Z}$.

Note that for an elliptic function $f(z)$, its behaviour on the fundamental parallelogram determines it for all $z \in \mathbb{C}$

z_1 in fundamental parallelogram

if $z_2 \equiv z_1$ modulo Λ_τ

then $f(z_2) = f(z_1)$



(19.2) If $f(z)$ is a holomorphic elliptic function then $f(z)$ is constant.

This is clear since $|f(z)|$ will attain absolute max. value on the fundamental parallelogram, since the latter is closed and bounded. But values of $f(z)$ just repeat on neighboring ~~rectang~~ parallelograms.

$\Rightarrow \exists\ M > 0$ such that $|f(z)| \leq M$ for every $z \in \mathbb{C}$

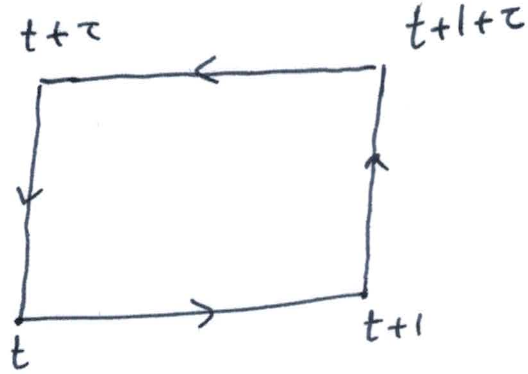
By Liouville's Theorem (Lecture 5, section (5.3)) f must be a constant.

(19.3) Let us take a non-constant elliptic function $f(z)$. Then number of poles of $f(z)$ within the fundamental parallelogram is finite.

This is because in the contrary case, the set of poles within the fundamental parallelogram will have a limit point (see Theorem (A.5) of Optional Reading I). But limit of poles is an essential singularity (see Lecture 11, section (11.5)) contradicting the hypothesis that $f(z)$ is meromorphic.

(19.4) Keeping f as before (a non constant elliptic function) number of zeroes of f within the fundamental parallelogram must also be finite. Since otherwise $\frac{1}{f}$ will have infinitely many poles contradicting (19.3).

Now let C be the simple closed curve:



where $t \in \mathbb{C}$ is chosen so that $f(z)$ has no zeroes or poles on C.

Such $t \in \mathbb{C}$ can always be found since number of poles and zeroes of f within the fundamental parallelogram is finite.

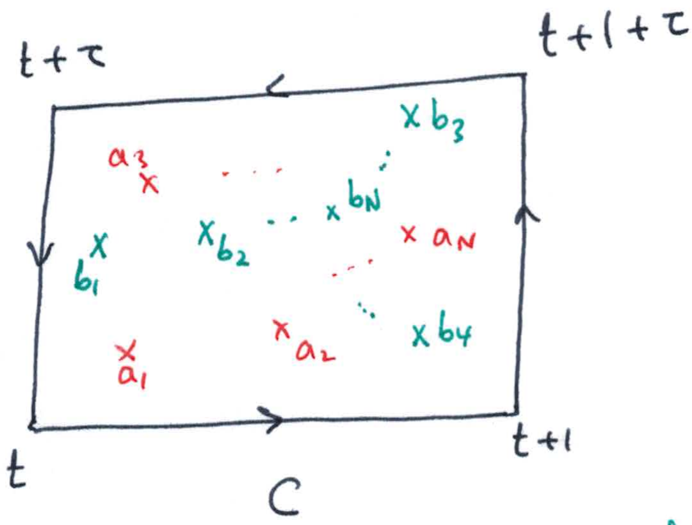
(19.5) **Theorem** ($f(z)$ and C as in (19.4) above).

(1) Sum of residues of $f(z)$ at poles within C $= 0$.

(2) Number of solutions of $f(z) = k$ ($k \in \mathbb{C}$ constant) within C $=$ Number of poles of $f(z)$ within C (is therefore same for all $k \in \mathbb{C}$)

(3) Let $a_1, \ldots, a_N$ be poles of $f(z)$ within C
$b_1, \ldots, b_N$ be zeroes of $f(z)$ within C
(listed according to multiplicity: i.e. if a is a pole of order n, then a appears in this list n times)

Then $$\sum_{\ell=1}^{N} a_\ell = \sum_{\ell=1}^{N} b_\ell \quad \text{modulo } \Lambda_\tau.$$



$b_1, b_2, \ldots, b_N \leftarrow$ zeroes of f
$a_1, a_2, \ldots, a_N \leftarrow$ poles of f

Proof

(1) Sum of residues

$$= \frac{1}{2\pi i} \int_C f(z)\,dz$$

$$= \frac{1}{2\pi i}\left[\int_t^{t+1} f(z)\,dz + \int_{t+1}^{t+1+\tau} f(z)\,dz + \int_{t+1+\tau}^{t+\tau} f(z)\,dz + \int_{t+\tau}^{t} f(z)\,dz \right]$$

By periodicity of f:

$$\int_{t+1}^{t+1+\tau} f(z)\,dz \underset{\text{set } z = w+1}{=} \int_{t}^{t+\tau} f(w+1)\,dw = \int_{t}^{t+\tau} f(w)\,dw = -\int_{t+\tau}^{t} f(w)\,dw$$

because $f(w+1) = f(w)$

Similarly $$\int_{t+1+\tau}^{t+\tau} f(z)\,dz = \int_{t+1}^{t} f(w+\tau)\,dw = -\int_{t}^{t+1} f(w)\,dw$$

Hence $$\frac{1}{2\pi i}\int_C f(z)\,dz = 0 \quad (= \text{sum of residues})$$

(2) Recall from Homework 5, problems (2) and (5)

(Number of solutions of $f(z) = k$) $-$ (Number of poles of $f(z)$)
(counted with multiplicity) (counted with order)

$$= \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-k}\,dz$$

Again using periodicity of $f(z)$ as in the previous part this integral is zero.

(3) $$\sum_{\ell=1}^{N} a_\ell - \sum_{\ell=1}^{N} b_\ell = \frac{1}{2\pi i}\int_C \frac{z\,f'(z)}{f(z)}\,dz \quad \text{(check this!)}$$

$$\frac{1}{2\pi i}\int_C \frac{z f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\left[\int_t^{t+1} - \int_{t+\tau}^{t+1+\tau} - \int_t^{t+\tau} + \int_{t+1}^{t+1+\tau}\right]$$

$$= \frac{1}{2\pi i}\left[\int_t^{t+1}\left(z\frac{f'(z)}{f(z)} - \frac{(z+\tau)f'(z+\tau)}{f(z+\tau)}\right)dz - \int_t^{t+\tau}\left(\frac{z f'(z)}{f(z)} - \frac{(z+1)f'(z+1)}{f(z+1)}\right)dz\right]$$

$$\underset{\text{(by periodicity of } f)}{=} \frac{1}{2\pi i}\left[\int_t^{t+\tau}\frac{f'(z)}{f(z)}\,dz - \tau\int_t^{t+1}\frac{f'(z)}{f(z)}\,dz\right]$$

$$= \frac{1}{2\pi i}\left(\Big[\log f(z)\Big]_{z=t}^{z=t+\tau} - \tau\Big[\log f(z)\Big]_{z=t}^{z=t+1}\right)$$

as $f(t) = f(t+\tau) = f(t+1)$, the values of $\log f(t)$ and $\log f(t+1)$ (or $\log f(t+\tau)$) differ by $2\pi i\cdot\mathbb{Z}$

$$\Rightarrow \quad \frac{1}{2\pi i}\int_C \frac{z f'(z)}{f(z)}\,dz \in \mathbb{Z} + \tau\mathbb{Z} = \Lambda_\tau .$$

(19.6) In order to construct examples of elliptic functions we need to first define Theta function.

Definition. $$\theta(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n e^{n(n-1)\pi i\tau} e^{2\pi i n z} \quad - (\star)$$

Let $q = e^{\pi i \tau}$. Since $\mathrm{Im}(\tau) > 0$, $|q| < 1$.

$$\theta(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} e^{2\pi i n z}$$

Theorem. The series on the right-hand side of $(\star)$ converges uniformly (on compact sets) and hence defines a holomorphic function on the entire complex plane ($\theta(z;\tau)$ has no poles). This function satisfies

$$\theta(z+1;\tau) = \theta(z;\tau) \qquad \theta(z+\tau;\tau) = -e^{-2\pi i z}\,\theta(z;\tau)$$

Moreover $\theta(z;\tau) = 0$ if and only if $z = m + n\tau \in \Lambda_\tau$ $(m, n \in \mathbb{Z})$ and each zero is of multiplicity 1.

Proof. Write
$$\theta(z;\tau) = 1 - e^{2\pi i z} + \sum_{n\geq 2} (-1)^n q^{n(n-1)} \left(e^{2\pi i n z} - e^{2\pi i(-n+1)z}\right)$$

$$\left(q^{n(n-1)} = q^{-n(-n+1)} = q^{m(m-1)} \quad \text{if } m = -n+1 \right)$$

if $z = x + iy$, $\left|e^{2\pi k i z}\right| = \left|e^{2\pi k i x}\right|\left|e^{-2\pi k y}\right|$

and $k \in \mathbb{Z}$ $\qquad = \left|e^{-2\pi y}\right|^{k}$

Therefore if $K \subset \mathbb{C}$ is a closed and bounded set, and we choose $A > 0$ such that $-\frac{A}{2\pi} \leq \operatorname{Im}(z) \leq \frac{A}{2\pi}$ for every $z \in K$

then for every $n \geq 1$, $\left|e^{2\pi n i z}\right| \leq e^{An}$

$$\left|e^{-2\pi n i z}\right| \leq e^{An}$$

$$\Rightarrow \left|e^{2\pi i n z} - e^{-2\pi i (n-1) z}\right| \leq 2e^{An} \qquad (\forall\ n \geq 2)$$

and the series $1 + e^{A} + 2\sum_{n \geq 2} q^{n(n-1)} e^{An}$ converges by ratio test:

Ratio of consecutive terms: $\left|\dfrac{q^{(n+1)(n)}\, e^{A(n+1)}}{q^{n(n-1)}\, e^{An}}\right| = |q|^{2n} e^{A} \to 0$ as $n \to \infty$ since $|q| < 1$.

Now clearly $\theta(z+1;\tau) = \theta(z;\tau)$ (since $e^{2\pi i n(z+1)} = e^{2\pi i n z}$ for every $n \in \mathbb{Z}$)

$$\theta(z+\tau;\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i \tau n(n-1)} e^{2\pi i n(z+\tau)}$$

$$= \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i \tau n(n+1)} e^{2\pi i n z}$$

$$= -e^{-2\pi i z} \sum_{n \in \mathbb{Z}} (-1)^{n+1} e^{\pi i \tau (n+1)(n+1-1)} e^{2\pi i (n+1) z} = -e^{-2\pi i z}\,\theta(z;\tau)$$

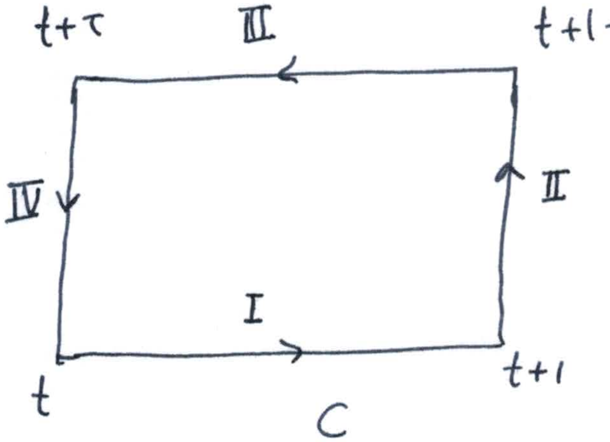
$$\theta(0;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} = 1 - 1 + \sum_{n\geq 2} (-1)^n q^{n(n-1)} (1-1) = 0$$

(see the first line of the proof)

$\Rightarrow \quad \theta(m+n\tau;\tau) = 0$ for every $m,n \in \mathbb{Z}$ by the periodicity properties proved above

We claim that number of zeroes of $\theta(z;\tau)$ within C $\longrightarrow$

$= 1$



Since θ has no poles, number of zeroes within C

$$= \frac{1}{2\pi i} \int_C \frac{\theta'(z;\tau)}{\theta(z;\tau)}\, dz .$$

Since $\theta(z+1;\tau) = \theta(z;\tau)$ the integral over $\underline{\mathrm{II}}$ cancels with integral over $\underline{\mathrm{IV}}$

Now
$$\frac{\theta'(z+\tau;\tau)}{\theta(z+\tau;\tau)} = \frac{\frac{d}{dz}\left(-e^{-2\pi i z}\theta(z;\tau)\right)}{-e^{-2\pi i z}\theta(z;\tau)} = -2\pi i + \frac{\theta'(z;\tau)}{\theta(z;\tau)}$$

$$\Rightarrow \frac{1}{2\pi i}\int_C \frac{\theta'(z;\tau)}{\theta(z;\tau)}\, dz = \frac{1}{2\pi i}\left[\int_t^{t+1} \frac{\theta'(z;\tau)}{\theta(z;\tau)}\, dz - \int_{t+\tau}^{t+1+\tau} \frac{\theta'(z;\tau)}{\theta(z;\tau)}\, dz\right]$$

$$= \frac{1}{2\pi i}\left[\int_t^{t+1} \frac{\theta'(z;\tau)}{\theta(z;\tau)}\, dz - \int_t^{t+1}\left(-2\pi i + \frac{\theta'(z;\tau)}{\theta(z;\tau)}\right) dz\right] = 1$$

as claimed

$\square$