Lecture 2

(2.0) Recall : We defined the notion of $\mathbb{C}$- differentiable function
$f: D \to \mathbb{C}$ where $D \subset \mathbb{C}$ is an open subset.

If $f(x+iy) = u(x,y) + i\,v(x,y)$ where u and v are two real-valued functions of 2 real variables, then

f is $\mathbb{C}$-differentiable iff u and v are differentiable and

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \text{Cauchy-Riemann equations.}$$

Alternate form of Cauchy - Riemann equations :

$$\begin{bmatrix} z = x + iy \\ \bar{z} = x - iy \end{bmatrix} \quad \text{equivalently} \quad \begin{bmatrix} x = \dfrac{z+\bar{z}}{2} \\ y = \dfrac{z-\bar{z}}{2i} \end{bmatrix}$$

Meaning we can think of f as function of 2 variables z and $\bar{z}$ instead of 2 variables x and y. Then

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} \quad \text{(Chain rule of partial derivatives)}$$

$$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{-1}{2i}\right) = \frac{1}{2} f_x + \frac{1}{2} i\, f_y$$

$$= \frac{1}{2}\left((u_x + i v_x) + i\,(u_y + i v_y)\right)$$

$$= \frac{1}{2}\left[(u_x - v_y) + i\,(v_x + u_y)\right]$$

Hence Cauchy - Riemann equations are equivalent to $\boxed{\dfrac{\partial f}{\partial \bar{z}} = 0}$

In other words f only depends on z and not on $\bar{z}$.

Summary :

$$\boxed{f \text{ is } \mathbb{C}\text{-differentiable}} \longleftrightarrow \boxed{\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}} \longleftrightarrow \boxed{\frac{\partial f}{\partial \bar{z}} = 0}$$

e.g. (i) Polynomial functions

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n \ ; \quad c_0, c_1, \dots, c_n \text{ are complex numbers}$$

are $\mathbb{C}$ - differentiable since $\frac{\partial f}{\partial \bar{z}} = 0$

$f'(z)$ is computed exactly in the same manner as real case

$$f'(z) = 0 + c_1 + 2c_2 z + 3c_3 z^2 + \dots + n c_n z^{n-1}$$

(ii) Rational functions : $f(z) = \frac{p(z)}{q(z)}$ where p and q are polynomial functions.

(2.1) The usual rules of differentiation hold in the complex case.

Namely,

(i) $$\frac{d}{dz}(z^n) = n z^{n-1} \ ; \quad n \text{ is an integer (positive)}$$

Proof: $$\frac{d}{dz}(z^n) = \lim_{h \to 0} \frac{(z+h)^n - z^n}{h}$$

$$(z+h)^n = z^n + n z^{n-1} h + \binom{n}{2} z^{n-2} h^2 + \binom{n}{3} z^{n-3} h^3 + \dots + n z h^{n-1} + h^n$$

(binomial expansion)

where $$\binom{n}{k} = \frac{n!}{(n-k)!\, k!}$$

$$\Rightarrow \lim_{h \to 0} \frac{(z+h)^n - z^n}{h} = \lim_{h \to 0} \frac{n z^{n-1} h + \binom{n}{2} z^{n-2} h^2 + \dots}{h}$$

$$= n z^{n-1}$$ $\square$

(ii) Product rule (or Leibniz rule):

$$\frac{d}{dz}\left(f(z)\cdot g(z)\right) = f'(z)\,g(z) + f(z)\,g'(z)$$

(iii) Chain rule:

$$\frac{d}{dz}\left(f(g(z))\right) = f'(g(z))\,g'(z)$$

Proof of product rule. Fix z in the domain of $f\cdot g$. We have to prove that given $\varepsilon > 0$, we can find $\delta > 0$ such that

$0 < |h| < \delta$ implies $\left| \dfrac{f(z+h)\,g(z+h) - f(z)\,g(z) - h\left(f'(z)\,g(z) + f(z)\,g'(z)\right)}{h} \right| < \varepsilon$

Numerator can be rewritten as

$$\left|\left(f(z+h) - f(z) - h\,f'(z)\right)\left(g(z+h) - g(z)\right) + g(z)\left(f(z+h) - f(z) - h\,f'(z)\right)\right.$$
$$\left. + f(z)\left(g(z+h) - g(z) - h\,g'(z)\right)\right|$$

$$\leq \left|f(z+h) - f(z) - h\,f'(z)\right|\left|g(z+h) - g(z)\right| + |g(z)|\left|f(z+h) - f(z) - h\,f'(z)\right|$$
$$+ |f(z)|\left|g(z+h) - g(z) - h\,g'(z)\right|$$

- Choose $\delta_1 > 0$ such that $\left|f(z+h) - f(z) - h\,f'(z)\right| < \dfrac{\varepsilon}{1 + |f(z)| + |g(z)|}|h|$ whenever $|h| < \delta_1$
- Choose $\delta_2 > 0$ such that $|g(z+h) - g(z)| < 1$ whenever $|h| < \delta_2$
- Choose $\delta_3 > 0$ such that $\left|g(z+h) - g(z) - h\,g'(z)\right| < \dfrac{\varepsilon}{1 + |f(z)| + |g(z)|}|h|$ whenever $|h| < \delta_3$

Take $\delta = \min(\delta_1, \delta_2, \delta_3)$ so that if $0 < |h| < \delta$

we get

$$\frac{1}{|h|}\left| f(z+h)g(z+h) - f(z)g(z) - h\left(f'(z)g(z) + f(z)g'(z)\right)\right|$$

$$\underset{*}{<} \frac{\varepsilon}{1+|f(z)|+|g(z)|}\left(1+|g(z)|+|f(z)|\right) = \varepsilon \quad \text{as desired} \quad \square$$

Proof of Chain rule. Again fix z in the domain. We have to prove that given $\varepsilon > 0$, we can find $\delta > 0$ such that $0 < |h| < \delta$ implies

$$\left| \frac{f(g(z+h)) - f(g(z))}{h} - f'(g(z))\,g'(z) \right| < \varepsilon$$

or equivalently $\quad \left| f(g(z+h)) - f(g(z)) - h\,f'(g(z))\,g'(z) \right| < \varepsilon |h|$

Let us rewrite this term as

$$\left| f(g(z+h)) - f(g(z)) - h\,f'(g(z))\,g'(z)\right|$$

$$= \left| f(g(z+h)) - f(g(z)) - h\left[\frac{g(z+h)-g(z)}{h}\right] f'(g(z)) + f'(g(z))\left(g(z+h) - g(z) - h\,g'(z)\right)\right|$$

$$\leq \left| f(g(z+h)) - f(g(z)) - h\left[\frac{g(z+h)-g(z)}{h}\right] f'(g(z))\right| + |f'(g(z))|\,|g(z+h) - g(z) - h\,g'(z)|$$

- Choose $\delta_1 > 0$ such that whenever $0 < |k| < \delta_1$ we have

(1) $\quad \left| f(g(z)+k) - f(g(z)) - k\,f'(g(z))\right| < \dfrac{\varepsilon\,|k|}{1+|g'(z)|+|f'(g(z))|}$

- Choose $\delta_2 > 0$ such that $|h| < \delta_2$ implies

  (2) $|g(z+h) - g(z)| < \delta_1$

- Choose $\delta_3 > 0$ such that $|h| < \delta_3$ implies

  (3) $|g(z+h) - g(z) - h\,g'(z)| < |h| \cdot \min\left(1, \dfrac{\varepsilon}{1+|g'(z)|+|f'(g(z))|}\right)$

Now take δ = smaller of δ_2, δ_3. Then for $|h| < \delta$ we have

$$|g(z+h) - g(z)| < \delta_1 \quad \text{(by (2))}$$

$$\Rightarrow \left| f(g(z) + (g(z+h) - g(z))) - f(g(z)) - (g(z+h) - g(z))\, f'(g(z)) \right|$$

(by (1))

$$< \frac{\varepsilon}{1+|g'(z)|+|f'(g(z))|} |g(z+h) - g(z)|$$

$$< \frac{\varepsilon}{1+|g'(z)|+|f'(g(z))|} |h| \left(1 + |g'(z)|\right)$$

The last step is by (3) since

$$|g(z+h) - g(z)| - |h|\,|g'(z)| \le |g(z+h) - g(z) - h g'(z)|$$

$$< |h|$$

$$\Rightarrow |g(z+h) - g(z)| < |h| \left(1 + |g'(z)|\right)$$

Therefore we get $|f(g(z+h)) - f(g(z)) - h\, f'(g(z))\, g'(z)|$

$$< \frac{\varepsilon\ |h|}{1+|g'(z)|+|f'(g(z))|} \left(1 + |g'(z)| + |f'(g(z))|\right) = \varepsilon |h|$$

as required. $\square$

(2.2) Another example. $f(z) = \frac{1}{z}$

$$f'(z) = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{z+h} - \frac{1}{z}\right] = \lim_{h\to 0} \frac{1}{h}\,\frac{z-z-h}{(z+h)h} = \lim_{h\to 0} \frac{-1}{(z+h)z}$$

$$= -\frac{1}{z^2}$$

This combined with chain rule and $\frac{d}{dz} z^n = n z^{n-1}$ ($n \geq 0$ integer)

implies that $\frac{d}{dz} z^{-n} = -n z^{-n-1}$.

(2.3) Integration along piece-wise smooth path.

For $f(z)$ a $\mathbb{C}$-differentiable function on an open set $D \subset \mathbb{C}$ and a piecewise smooth path γ in D, we wish to define

$$\int_\gamma f(z)\,dz \qquad (= \text{a complex number})$$

We begin by defining the notion of a piecewise smooth path (or curve).

Definition. A function $\gamma : [a,b] \to \mathbb{C}$ will be called a path (or curve) in $\mathbb{C}$.

Continuity: for every $t_0 \in (a,b)$ $\lim_{t\to t_0} \gamma(t) = \gamma(t_0)$ and

$$\lim_{t\to a^+} \gamma(t) = \gamma(a) \quad \text{and} \quad \lim_{t\to b^-} \gamma(t) = \gamma(b)$$

Such paths are called <u>continuous</u>.

Differentiability: for every $t_0 \in (a,b)$, ~~$\lim_{t\to t_0}\gamma$~~

$$\lim_{h\to 0} \frac{\gamma(t_0+h) - \gamma(t_0)}{h} \quad \text{exists.} \quad (=: \gamma'(t_0))$$

Note: These notions are nothing new. $\gamma(t) = x(t) + i\,y(t)$ was termed a parametric curve in Calculus III. Continuity and differentiability of γ are just the same properties of $x(t)$ and $y(t)$.

That is, $\gamma'(t) = x'(t) + i\,y'(t)$.

Definition. A smooth path in $\mathbb{C}$ is an everywhere differentiable function $\gamma : [a,b] \longrightarrow \mathbb{C}$.

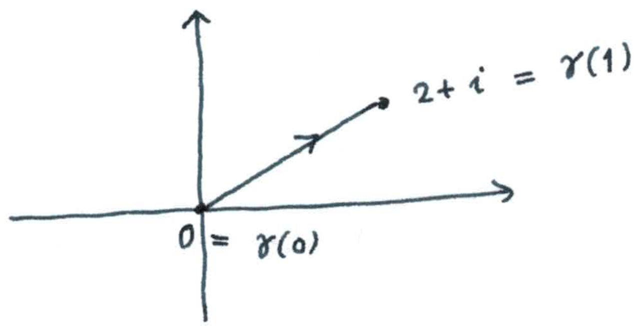
More generally, we say γ is piecewise smooth if we can partition $[a,b]$ into finitely many subintervals

$$t_0 = a < t_1 < \cdots < t_{n-1} < t_n = b$$

such that the restriction of γ to each of the subinterval is smooth. That is, γ is smooth on $[t_0,t_1], [t_1,t_2], \ldots, [t_{n-1},t_n]$.

e.g. (i) $\gamma(t) = t(2+i)$, $0 \le t \le 1$ is a straight line joining 0 and $2+i$

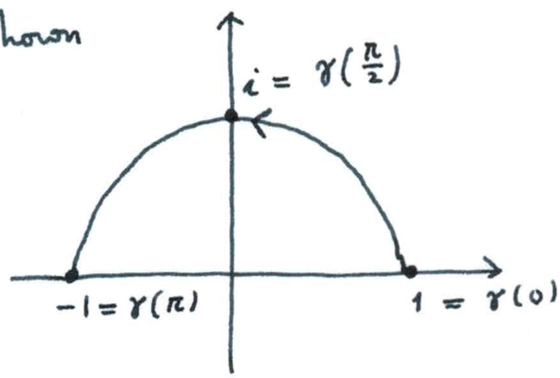
$$[0,1] \xrightarrow{\ \gamma\ }$$



This is a smooth path.

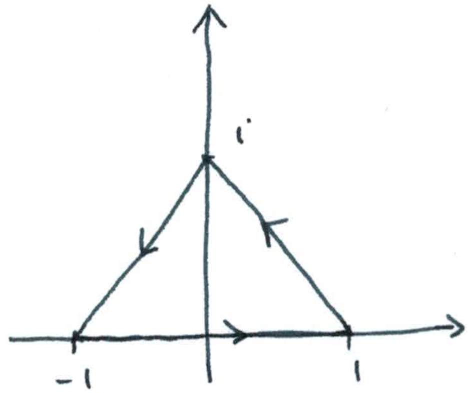
(ii) $\gamma(t) = \cos(t) + i\sin(t)$ $\quad 0 \le t \le \pi$

= semi circle of radius 1 as shown is a smooth path.



(iii) $\gamma(t)$ describing the boundry of a triangle with vertices $1,\ i,\ -1$ as shown:

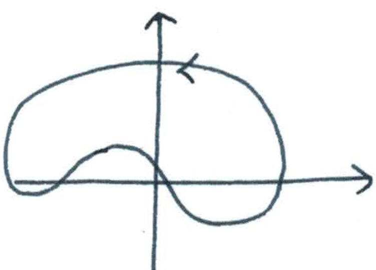
$$\gamma(t) = \begin{cases} -1+2t\ ; & 0 \le t \le 1 \\ (2-t)+(t-1)i\ ; & 1 \le t \le 2 \\ (2-t)+(3-t)i\ ; & 2 \le t \le 3 \end{cases}$$



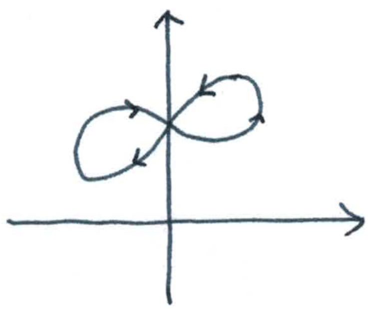
is not smooth, but piecewise smooth

(2.4) In this course we will only deal with piecewise smooth paths $\gamma : [a,b] \to \mathbb{C}$. So most of the time the adjective "piecewise smooth" will be omitted.

Last of all, we say γ is <u>closed</u> if $\gamma(a) = \gamma(b)$. We say γ is <u>simple</u> if for any $t_1 \neq t_2$ $(a < t_1 < b,\ a < t_2 < b)$ $\gamma(t_1) \neq \gamma(t_2)$ (the path does not intersect itself except possibly at the end point(s)).



a simple closed path



NOT simple but closed