Lecture 20

(20.0) Recall: we defined a holomorphic function $\theta(z;\tau)$ for a fixed complex number $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) > 0$:

$$\theta(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i \tau n(n-1)} e^{2\pi i n z}$$

$$= \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} e^{2\pi i n z} \quad \text{where } q = e^{\pi i \tau}$$

(since $\mathrm{Im}(\tau) > 0$, $|q| < 1$). We proved that

(1) $\theta(z;\tau)$ is a holomorphic function of $z \in \mathbb{C}$. It only has zeroes (of multiplicity 1) at $z = m + n\tau$ $(m,n\in\mathbb{Z})$

Notation $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\cdot\tau \subset \mathbb{C}$ called <u>lattice</u> (of rank 2) in $\mathbb{C}$

(2) $\theta(z+1;\tau) = \theta(z;\tau)$

$\theta(z+\tau;\tau) = -e^{-2\pi i z}\,\theta(z;\tau)$

(20.1) A differential equation satisfied by $\theta(z;\tau)$

$$\frac{\partial}{\partial z}\theta(z,\tau) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i \tau n(n-1)} (2\pi i n)\, e^{2\pi i n z} \quad - \text{①}$$

$$\frac{\partial^2}{\partial z^2}\theta(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n(n-1)\tau} (2\pi i n)^2 e^{2\pi i n z} \quad -(2)$$

$$\frac{\partial}{\partial \tau}\theta(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n \left(\pi i (n^2-n)\right) e^{\pi i \tau n(n-1)} e^{2\pi i n z} \quad -(3)$$

We get $$\frac{1}{\pi i}\frac{\partial}{\partial\tau}\theta(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} n^2 e^{2\pi i n z}$$

$$- \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} \cdot n\, e^{2\pi i n z} \quad (\text{from } (3))$$

$$= \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}\theta(z;\tau) - \frac{1}{2\pi i}\frac{\partial}{\partial z}\theta(z;\tau) \quad (\text{from } (1) \text{ and } (2))$$

Heat equation: $$\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}\theta(z;\tau) - \frac{1}{2\pi i}\frac{\partial}{\partial z}\theta(z;\tau) = \frac{1}{\pi i}\frac{\partial}{\partial \tau}\theta(z;\tau)$$

(20.2) Product formula of Jacobi – some variants of theta:

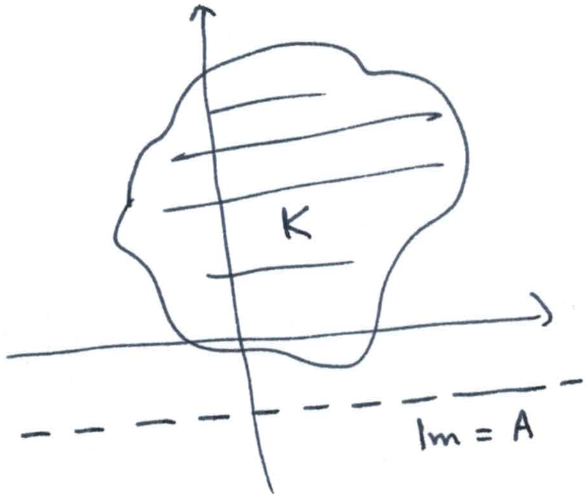
Consider the infinite product $\prod_{n=0}^{\infty}\left(1-q^{2n}e^{2\pi i z}\right)$

__Theorem.__ $\prod_{n=0}^{\infty}\left(1-q^{2n}e^{2\pi i z}\right)$ converges uniformly (on compact subsets of $\mathbb{C}$)

Proof. Recall that if $z = x+iy$, $e^{2\pi i z} = e^{2\pi i x} e^{-2\pi y}$

So $\left| e^{2\pi i z} \right| = e^{-2\pi y}$ $\quad (y = \mathrm{Im}(z))$.

Given a compact set $K \subset \mathbb{C}$, let $A \in \mathbb{R}$ be such that $\mathrm{Im}(z) \geq A$ for every $z \in K$, so that $\left| e^{2\pi i z} \right| \leq e^{-2\pi A}$ for every $z \in K$.



For convenience, write $w = e^{2\pi i z}$

Since $|q| < 1$, $q^n \to 0$ as $n \to \infty$

Pick $N > 0$ large enough so that $|q|^{2n} e^{-2\pi A} < \frac{1}{2}$ (for $n \geq N$).

Then for every $z \in K$, $n \geq N$ we get $\left(w = e^{2\pi i z} \right)$

$$\left| \log\left(1 - q^{2n} w\right) \right| = \left| q^{2n} w + \frac{(q^{2n} w)^2}{2} + \frac{(q^{2n} w)^3}{3} + \dots \right|$$

$$= |q|^{2n} |w| \cdot \left\{ \left| 1 + \frac{q^{2n} w}{2} + \frac{(q^{2n} w)^2}{3} + \dots \right| \right\}$$

$$\leq e^{-2\pi A} |q|^{2n} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right) = 2 e^{-2\pi A} |q|^n$$

So $\left| \sum_{n=N}^{\infty} \log\left(1 - q^n w\right) \right| \leq 2 e^{-2\pi A} \sum_{n=N}^{\infty} |q|^{2n}$ converges (as $|q|^2 < 1$)

and hence so does $\prod_{n=N}^{\infty} \left(1 - q^{2n} w\right)$ $\square$

(20.3) Let $\theta_+(z;\tau) = \prod_{n=0}^{\infty}(1-q^{2n}e^{2\pi i z})$ (recall: $q = e^{\pi i \tau}$)

Then $\theta_+(z;\tau)$ is a holomorphic function of $z \in \mathbb{C}$ by Theorem (20.2). Moreover, $\theta_+(z;\tau) = 0$ if and only if

$$e^{2\pi i z} = q^{2n} = e^{2\pi i n\tau} \quad \text{for some } n = 0, 1, 2, \dots$$

$$\equiv \quad z = m + n\tau \quad \text{for some } \begin{array}{l} m \in \mathbb{Z} \\ n \in \mathbb{N} = \{0,1,2,\dots\} \end{array}$$

(each with mult. 1)

Moreover $\theta_+(z+1;\tau) = \theta_+(z;\tau)$

$$\theta_+(z+\tau;\tau) = \prod_{n=0}^{\infty}(1-q^{2n}q^2e^{2\pi i z})$$

$$= \prod_{n=1}^{\infty}(1-q^{2n}e^{2\pi i z}) = \frac{\theta_+(z;\tau)}{1-e^{2\pi i z}}$$

(20.4) Similarly define $\theta_-(z;\tau) = \prod_{n=1}^{\infty}(1-q^{2n}e^{-2\pi i z})$

Check: • $\theta_-(z;\tau) = 0$ if and only if $z = m - n\tau$ $\left(\begin{array}{l} m \in \mathbb{Z} \\ n = 1, 2, 3, \dots \end{array}\right)$

(each with mult. 1)

• $\theta_-(z+1;\tau) = \theta_-(z;\tau)$

$$\theta_-(z+\tau;\tau) = (1-e^{-2\pi i z})\,\theta_-(z;\tau)$$

So $f(z) = \theta_+(z;\tau)\,\theta_-(z;\tau)$ satisfies

- $f(z) = 0 \iff z = m + n\tau \quad (m, n \in \mathbb{Z})$ each with multiplicity 1
- $f(z+1) = f(z)$
- $f(z+\tau) = \theta_+(z+\tau;\tau)\,\theta_-(z+1;\tau)$

$$= \frac{1 - e^{-2\pi i z}}{1 - e^{2\pi i z}}\,\theta_+(z;\tau)\,\theta_-(z;\tau) = -e^{-2\pi i z} f(z)$$

exact same properties of $\theta(z;\tau)$. So, if we conside the ratio $\dfrac{\theta(z;\tau)}{f(z)}$ it is a holomorphic function which is elliptic. But by (19.2) of Lecture 19, any such function has to be constant : $\theta(z;\tau) = G \cdot f(z)$ for some constant G. Summarizing,

$$\theta(z;\tau) = G \cdot \theta_+(z;\tau)\ \theta_-(z;\tau)$$

$$= G \cdot \prod_{n=0}^{\infty}\left(1 - q^{2n} e^{2\pi i z}\right) \cdot \prod_{n=1}^{\infty}\left(1 - q^{2n} e^{-2\pi i z}\right)$$

$G = G(\tau)$ is a number (depending only on τ and not on z).

(20.5) $$\theta(z;\tau) = G\cdot(1-e^{2\pi i z})\prod_{n=1}^{\infty} 1-q^{2n}e^{2\pi i z}\prod_{n=1}^{\infty} 1-q^{2n}e^{-2\pi i z}$$

$$= G\,(1-e^{2\pi i z}).\ T(z) \quad \text{where } T(z) = \prod_{n=1}^{\infty} 1-q^{2n}w\prod_{n=1}^{\infty} 1-q^{2n}w^{-1}$$

$(w = e^{2\pi i z})$

So we get $\left(\theta' = \frac{\partial}{\partial z}\theta \text{ below}\right)$

$$\theta'(z;\tau) = G\cdot\left[\left(-2\pi i\, e^{2\pi i z}\right)T(z) + \left(1-e^{2\pi i z}\right)T'(z)\right]$$

$$\theta''(z;\tau) = G\cdot\left[-(2\pi i)^2 e^{2\pi i z}\,T(z) - 4\pi i\, e^{2\pi i z}\,T'(z) + \left(1-e^{2\pi i z}\right)T''(z)\right]$$

$$\theta'''(z;\tau) = G\cdot\left[-(2\pi i)^3 e^{2\pi i z}\,T(z) - 3(2\pi i)^2 e^{2\pi i z}\,T'(z) - 3(2\pi i)\,e^{2\pi i z}\,T''(z) + \left(1-e^{2\pi i z}\right)T'''(z)\right]$$

and so on.

Using the Heat equation from section (20.1) we get

$$\frac{1}{\pi i}\frac{\partial}{\partial \tau}\theta'(z;\tau) = \frac{1}{(2\pi i)^2}\theta'''(z;\tau) - \frac{1}{2\pi i}\theta''(z;\tau) \quad — (\star)$$

At $z=0$, the formulae for the derivatives above imply

$$\theta'(0;\tau) = (-2\pi i)\,G.T(0)$$

$$\theta''(0;\tau) = G\left[-(2\pi i)^2\,T(0) - 4\pi i\,T'(0)\right]$$

$$\theta'''(0;\tau) = G\left[-(2\pi i)^3\,T(0) - 3(2\pi i)^2\,T'(0) - 3(2\pi i)\,T''(0)\right]$$

Therefore letting $z=0$ in (✡) we get

$$\frac{1}{\pi i}\frac{\partial}{\partial\tau}\theta'(0;\tau) = \frac{G}{(2\pi i)^2}\left[\cancel{-(2\pi i)^3 T(0)} - 3(2\pi i)^2 T'(0) - 3(2\pi i)T''(0)\right]$$

$$-\frac{G}{2\pi i}\left[\cancel{-(2\pi i)^2 T(0)} - 2(2\pi i)T'(0)\right]$$

$$= G\left[-T'(0) - \frac{3}{2\pi i}T''(0)\right]$$

So $$\frac{1}{\pi i}\left[\frac{1}{\theta'(0;\tau)}\frac{\partial}{\partial\tau}\theta'(0;\tau)\right] = \frac{1}{-2\pi i\,\cancel{G}\cdot T(0)}\cancel{G}\left(-T'(0)-\frac{3}{2\pi i}T''(0)\right)$$

$$= \frac{1}{2\pi i}\frac{T'(0)}{T(0)} + \frac{3}{(2\pi i)^2}\frac{T''(0)}{T(0)}$$

(20.6) Now let us compute $\dfrac{T'(0)}{T(0)}$ and $\dfrac{T''(0)}{T(0)}$

Since $T(z) = \prod_{n=1}^{\infty} 1-q^{2n}e^{2\pi i z}\prod_{n=1}^{\infty} 1-q^{2n}e^{-2\pi i z}$, we get

$$\frac{T'(z)}{T(z)} = \sum_{n=1}^{\infty}\frac{-(2\pi i)q^{2n}e^{2\pi i z}}{1-q^{2n}e^{2\pi i z}} + \sum_{n=1}^{\infty}\frac{-(-2\pi i)q^{2n}e^{-2\pi i z}}{1-q^{2n}e^{-2\pi i z}} \qquad (4)$$

(this is just logarithmic derivative).

Take $\frac{d}{dz}$ of (4), and note that $\frac{d}{dz}\left(\frac{q^{2n}e^{2\pi i z}}{1-q^{2n}e^{2\pi i z}}\right) = \frac{d}{dz}\left[-1+\frac{1}{1-q^{2n}e^{2\pi i z}}\right]$

$$= \frac{-1}{(1-q^{2n}e^{2\pi i z})^2}\cdot\left(-q^{2n}(2\pi i)e^{2\pi i z}\right)$$

we get

$$\frac{T''(z)}{T(z)} - \frac{T'(z)^2}{T(z)^2} = \sum_{n=1}^{\infty} \frac{-(2\pi i)^2 q^{2n} e^{2\pi i z}}{(1-q^{2n}e^{2\pi i z})^2} + \sum_{n=1}^{\infty} \frac{-(2\pi i)^2 q^{2n} e^{-2\pi i z}}{(1-q^{2n}e^{-2\pi i z})^2} \quad — (5)$$

$$\left(\frac{d}{dz}\left(\frac{T'(z)}{T(z)}\right)\right)$$

From (4) $\frac{T'(0)}{T(0)} = 0$ and from (5) $\frac{T''(0)}{T(0)} = -2(2\pi i)^2 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2}$

Combining with the calculation of (20.5) we get

$$\boxed{\frac{1}{\pi i}\left[\frac{1}{\theta'(0;\tau)} \frac{\partial}{\partial \tau} \theta'(0;\tau)\right] = -6 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2}}$$

(20.7) Other conventions for theta functions.

Recall that $\sin(x)$ has periodicity $\sin(x+2\pi) = \sin(x)$

$\sin(x+\pi) = -\sin(x)$

we can define $\cos(x)$ as half shift relative to period π

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

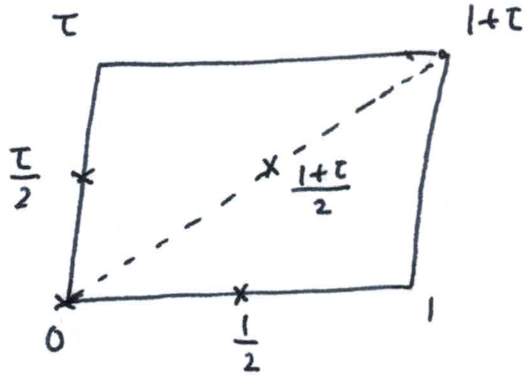
In our situation the periods are 1 and τ. So, we can define 3 more functions in this spirit.

$\theta_1(z;\tau) = \theta(z;\tau)$ as before

$\theta_2(z;\tau) = \theta(z+\tfrac{1}{2};\tau)$, $\theta_3(z;\tau) = \theta(z+\tfrac{1}{2}+\tfrac{\tau}{2};\tau)$ and

$\theta_4(z;\tau) = \theta(z+\tfrac{\tau}{2};\tau)$

All the computations given above can be performed for $\theta_{2,3,4}$ as well:



$$\theta_2(z;\tau) = \sum_{n\in\mathbb{Z}} q^{n(n-1)} w^n = G \cdot \prod_{n=0}^{\infty} 1+q^{2n}w \prod_{n=1}^{\infty} 1+q^{2n}w^{-1}$$

satisfies the same heat equation $\frac{1}{\pi i}\frac{\partial}{\partial\tau} = \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2} - \frac{1}{2\pi i}\frac{\partial}{\partial z}$

Analog of the result of (20.6) is (homework)

$$\frac{1}{\pi i}\left[\frac{1}{\theta_2(0;\tau)}\frac{\partial}{\partial\tau}\theta_2(0;\tau)\right] = 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2}$$

$$\theta_3(z;\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} w^n = G \cdot \prod_{n=0}^{\infty} 1+q^{2n+1}w \prod_{n=1}^{\infty} 1+q^{2n-1}w^{-1}$$

satisfies a variant of the heat equation: $\boxed{\frac{1}{\pi i}\frac{\partial}{\partial\tau} = \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}}$

Analog of the computation given in (20.6):

$$\frac{1}{\pi i}\left[\frac{1}{\theta_3(0;\tau)}\frac{\partial}{\partial \tau}\theta_3(0;\tau)\right] = 2\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1+q^{2n-1})^2}$$

$$\theta_4(z;\tau) = \sum_{n\in\mathbb{Z}}(-1)^n q^{2n^2}\omega^n = G\prod_{n\geq 0} 1-q^{2n+1}\omega \prod_{n\geq 1} 1-q^{2n-1}\omega^{-1}$$

satisfies $\frac{1}{\pi i}\frac{\partial}{\partial \tau} = \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}$ and we have

$$\frac{1}{\pi i}\left[\frac{1}{\theta_4(0;\tau)}\frac{\partial}{\partial \tau}\theta_4(0;\tau)\right] = -2\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}$$