Lecture 21

(21.0) Recall : the summary of 4 theta functions

- Series and product expression:

$$\theta_1(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} w^n = G \cdot \prod_{n\geq 0}(1-q^{2n}w) \prod_{n\geq 1}(1-q^{2n}w^{-1})$$

$$\theta_2(z;\tau) = \sum_{n\in\mathbb{Z}} q^{n(n-1)} w^n = G \cdot \prod_{n\geq 0}(1+q^{2n}w) \prod_{n\geq 1}(1+q^{2n}w^{-1})$$

$$\theta_3(z;\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} w^n = G \cdot \prod_{n\geq 0}(1+q^{2n+1}w) \prod_{n\geq 1}(1+q^{2n-1}w^{-1})$$

$$\theta_4(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} w^n = G \cdot \prod_{n\geq 0} 1-q^{2n+1}w \prod_{n\geq 1}(1-q^{2n-1}w^{-1})$$

Recall: $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau)>0$. $q = e^{\pi i \tau}$ $(|q|<1)$

$w = e^{2\pi i z}$ (for convenience only)

- Heat equation. θ_1 and θ_2 satisfy

$$\frac{1}{\pi i}\frac{\partial}{\partial \tau} = \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2} - \frac{1}{2\pi i}\frac{\partial}{\partial z}$$

θ_3 and θ_4 satisfy : $\displaystyle \frac{1}{\pi i}\frac{\partial}{\partial \tau} = \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}$

- Long computation using the heat equation yields the following :

$$\frac{1}{\pi i}\left[\frac{1}{\theta_1'(0;\tau)}\frac{\partial}{\partial\tau}\theta_1'(0;\tau)\right] = -6\sum_{n\geq 1}\frac{q^{2n}}{(1-q^{2n})^2}$$

$$\frac{1}{\pi i}\left[\frac{1}{\theta_2(0;\tau)}\frac{\partial}{\partial\tau}\theta_2(0;\tau)\right] = 2\sum_{n\geq 1}\frac{q^{2n}}{(1+q^{2n})^2}$$

$$\frac{1}{\pi i}\left[\frac{1}{\theta_3(0;\tau)}\frac{\partial}{\partial\tau}\theta_3(0;\tau)\right] = 2\sum_{n\geq 1}\frac{q^{2n-1}}{(1+q^{2n-1})^2}$$

$$\frac{1}{\pi i}\left[\frac{1}{\theta_4(0;\tau)}\frac{\partial}{\partial\tau}\theta_4(0;\tau)\right] = -2\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}$$

(21.1)

$$\theta_1'(0;\tau) = -\pi i\,\theta_2(0;\tau)\,\theta_3(0;\tau)\,\theta_4(0;\tau)$$

Proof. Note that $\displaystyle\sum_{n\geq 1}\frac{q^{2n}}{(1+q^{2n})^2} - \sum_{n\geq 1}\frac{q^{2n}}{(1-q^{2n})^2} = -4\sum_{n\geq 1}\frac{q^{4n}}{(1-q^{4n})^2}$

and $$\sum_{n\geq 1}\frac{q^{2n-1}}{(1+q^{2n-1})^2} - \sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2} = -4\sum_{n\geq 1}\frac{q^{2(2n-1)}}{(1-q^{2(2n-1)})^2}$$

Adding these

$$\sum_{n\geq 1}\frac{q^{2n}}{(1+q^{2n})^2} + \sum_{n\geq 1}\frac{q^{2n-1}}{(1+q^{2n-1})^2} - \sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2} = -3\sum_{n\geq 1}\frac{q^{2n}}{(1-q^{2n})^2}$$

$\Rightarrow$

$$-6\sum_{n\geq 1}\frac{q^{2n}}{(1-q^{2n})^2} = 2\sum_{n\geq 1}\frac{q^{2n}}{(1+q^{2n})^2} + 2\sum_{n\geq 1}\frac{q^{2n-1}}{(1+q^{2n-1})^2} - 2\sum_{n\geq 1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}$$

$$\Rightarrow \quad \frac{1}{\theta_1'(0;\tau)}\frac{\partial}{\partial\tau}\theta_1'(0;\tau) = \sum_{j=2}^{4}\frac{1}{\theta_j(0;\tau)}\frac{\partial}{\partial\tau}\theta_j(0;\tau)$$

This is just logarithmic derivative (with respect to τ). Hence

$$\theta_1'(0;\tau) = C\,\theta_2(0;\tau)\,\theta_3(0;\tau)\,\theta_4(0;\tau)$$

where $C\in\mathbb{C}$ is a constant. To get the value of C, let $\operatorname{Im}(\tau)\to\infty$ (or equivalently $q=e^{\pi i\tau}\to 0$). Then (using the infinite summation formulae for these theta functions):

$$\theta_2(z;\tau) \longrightarrow 1+e^{2\pi i z} \ \underline{\underline{\text{At } (z=0)}}\ \ 2$$

$$\theta_3(z;\tau) \longrightarrow 1$$

$$\theta_4(z;\tau) \longrightarrow 1 \qquad \text{.}$$

(as $q\to 0$)

Finally (from last lecture)

$$\theta_1'(0;\tau) \longrightarrow -2\pi i \quad \text{as} \quad q\to 0$$

Hence $\quad C = -\pi i \quad$ as claimed

$\square$

(21.2) $$G = \prod_{n\geq 1}(1-q^{2n})$$

Proof. From $\theta_1'(0;\tau) = -\pi i\, \theta_2(0;\tau)\, \theta_3(0;\tau)\, \theta_4(0;\tau)$ and the infinite product expansion of these functions we get

$$-2\pi i G \prod_{n\geq 1}(1-q^{2n})^2 = -2\pi i G^3 \prod_{n\geq 1}(1+q^{2n})^2 \prod_{n\geq 1}(1+q^{2n-1})^2 \prod_{n\geq 1}(1-q^{2n-1})^2$$

$$\Rightarrow \quad G^2 = \frac{\prod_{n\geq 1}(1-q^{2n})^2}{\prod_{n\geq 1}(1+q^{2n})^2 \prod_{n\geq 1}(1-q^{2(2n-1)})^2} = \frac{\prod_{n\geq 1}(1-q^{2(2n)})^2}{\prod_{n\geq 1}(1+q^{2n})^2}$$

$$= \prod_{n\geq 1}(1-q^{2n})^2$$

So $$G = \pm \prod_{n\geq 1}(1-q^{2n})^2$$

But $$\lim_{q\to 0} \theta_1'(0;\tau) = -2\pi i \lim_{q\to 0} G(q) \quad \text{(from product expression)}$$

$$= -2\pi i \quad \text{(from sum expression)}$$

$\Rightarrow \lim_{q\to 0} G(q) = 1$ hence $+$ sign must be taken $\square$

(21.3) We have proved that ($\theta_1 = \theta$ recall)

$$\theta(z;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} e^{2\pi i n z} \quad \text{(definition)}$$

$$= \prod_{n\geq 1}(1-q^{2n}) \prod_{n\geq 0} 1-q^{2n} e^{2\pi i z} \prod_{n\geq 1}(1-q^{2n} e^{-2\pi i z})$$

called Jacobi's triple product identity

e.g. $\theta(-z) = -e^{-2\pi i z}\,\theta(z)$

Since $\theta(z) = (1-e^{2\pi i z}) \underbrace{\prod_{n\geq 1} 1-q^{2n} \prod_{n\geq 1} 1-q^{2n} e^{2\pi i z} \prod_{n\geq 1} 1-q^{2n} e^{-2\pi i z}}_{\text{symmetric (i.e. does not change if } z \leftrightarrow -z)}$

So $\theta(-z) = (1-e^{-2\pi i z}) \prod_{n\geq 1} 1-q^{2n} \prod_{n\geq 1} 1-q^{2n} e^{2\pi i z} \prod_{n\geq 1} 1-q^{2n} e^{-2\pi i z}$

$= -e^{-2\pi i z}\,\theta(z)$ as claimed $\square$

There are several identities one could write down among theta functions. Their proof essentially boils down to the fact that

$$\boxed{\text{There are no non-constant holomorphic elliptic functions}}$$

(21.4) Fay's Trisecant Identity.

Let $\alpha, \beta, \gamma \in \mathbb{C}$. Consider the following function

$$f(z) = \frac{\theta(z-\beta)\,\theta(z+\beta)\,\theta(\alpha-\gamma)\,\theta(\alpha+\gamma) - \theta(z-\gamma)\,\theta(z+\gamma)\,\theta(\alpha-\beta)\,\theta(\alpha+\beta)}{\theta(z-\alpha)\,\theta(z+\alpha)\,\theta(\beta-\gamma)\,\theta(\beta+\gamma)}$$

Ex. <u>f is elliptic.</u> $f(z+1) = f(z)$ is clear since it is true for θ.

$f(z+\tau) = f(z)$ since we get $e^{-4\pi i z}$ both in numerator and denominator and these factors cancel each other. (Recall: $\theta(z+\tau) = -e^{-2\pi i z}\,\theta(z)$)

<u>f has no poles.</u> The only apparent poles of $f(z)$ are when the denominator equals zero. Recall $\theta(x) = 0$ if and only if $x = 0$ (modulo Λ_τ) with multiplicity 1.

$$z = \alpha: \quad \text{Numerator} = \theta(\alpha-\beta)\,\theta(\alpha+\beta)\,\theta(\alpha-\gamma)\,\theta(\alpha+\gamma) - \theta(\alpha-\gamma)\,\theta(\alpha+\gamma)\,\theta(\alpha-\beta)\,\theta(\alpha+\beta) = 0$$

$$z = -\alpha: \quad \text{Numerator} = \theta(-\alpha-\beta)\,\theta(-\alpha+\beta)\,\theta(\alpha-\gamma)\,\theta(\alpha+\gamma) - \theta(-\alpha-\gamma)\,\theta(-\alpha+\gamma)\,\theta(\alpha-\beta)\,\theta(\alpha+\beta)$$

$$\underset{\text{(previous section)}}{=} e^{-2\pi i(\alpha+\beta+\alpha-\beta)}\,\theta(\alpha-\beta)\,\theta(\alpha+\beta)\,\theta(\alpha-\gamma)\,\theta(\alpha+\gamma) - e^{-2\pi i(\alpha+\gamma+\alpha-\gamma)}\,\theta(\alpha+\gamma)\,\theta(\alpha-\gamma)\,\theta(\alpha-\beta)\,\theta(\alpha+\beta) = 0$$

$\Rightarrow$ $f(z)$ is holomorphic and elliptic. So it must be constant. To determine the value of this constant we can set $z =$ any value we like.

e.g (assume $\beta \neq \alpha$) set $z = \beta$. We get

$$f(z) = \frac{\theta(0)\,\theta(2\beta)\,\theta(\alpha-\gamma)\,\theta(\alpha+\gamma) - \theta(\beta-\gamma)\,\theta(\beta+\gamma)\,\theta(\alpha-\beta)\,\theta(\alpha+\beta)}{\theta(\beta-\alpha)\,\theta(\beta+\alpha)\,\theta(\beta-\gamma)\,\theta(\beta+\gamma)}$$

$$= -\frac{\theta(\alpha-\beta)\,\theta(\alpha+\beta)}{\theta(\beta-\alpha)\,\theta(\beta+\alpha)} \qquad (\text{since } \theta(0)=0)$$

$$= -\left(-e^{-2\pi i(\beta-\alpha)}\right)\frac{\theta(\beta-\alpha)\,\theta(\alpha+\beta)}{\theta(\beta-\alpha)\,\theta(\beta+\alpha)} = e^{2\pi i(\alpha-\beta)}$$

Hence we obtain:

$$\theta(z-\beta)\,\theta(z+\beta)\,\theta(\alpha-\gamma)\,\theta(\alpha+\gamma) - \theta(z-\gamma)\,\theta(z+\gamma)\,\theta(\alpha-\beta)\,\theta(\alpha+\beta) = e^{2\pi i(\alpha-\beta)}\left(\theta(z-\alpha)\,\theta(z+\alpha)\,\theta(\beta-\gamma)\,\theta(\beta+\gamma)\right)$$

[Trisecant Identity].