Lecture 22

Recall that we were studying properties of theta function

(22.0) $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$, $q = e^{\pi i \tau}$

$\Lambda = \{ m + n\tau$ where $m, n \in \mathbb{Z} \} \subset \mathbb{C}$

$$\theta(z;\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(n-1)} e^{2\pi i n z}$$

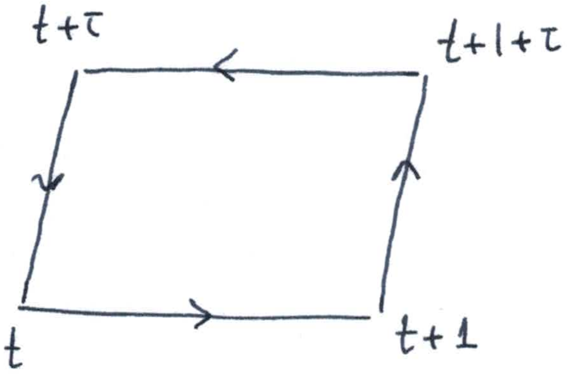
$$= \prod_{n \geq 1} 1 - q^{2n} \prod_{n \geq 0} 1 - q^{2n} e^{2\pi i z} \prod_{n \geq 1} 1 - q^{2n} e^{-2\pi i z}$$

(Jacobi's triple product formula)

- $\theta(z;\tau)$ is a holomorphic function of $z \in \mathbb{C}$, $(\tau \in \mathbb{C},\ \mathrm{Im}(\tau) > 0)$
- $\theta(z+1;\tau) = \theta(z;\tau)$
  $\theta(z+\tau;\tau) = -e^{-2\pi i z}\theta(z;\tau)$
- $\theta(z;\tau) = 0$ if and only if $z \in \Lambda$
  (each with multiplicity 1)

(22.1) Any elliptic function can be written as a rational expression in terms of the theta function.

Proof. Let $f(z)$ be an arbitrary non-constant elliptic function. Let $t \in \mathbb{C}$ be such that $f(z)$ does not have any zeroes or poles on C :



(see Lecture 19, sections (19.4) and (19.5))

We know that

$$\#\text{ of zeroes of } f \text{ within } C = \#\text{ of poles of } f \text{ within } C$$

(counted with mult.) (counted with order)

Let $a_1, a_2, \ldots, a_N$ = zeroes of f within C

$b_1, b_2, \ldots, b_N$ = poles of f within C

(listed according to their multiplicity : i.e., a pole of order 5 appears 5 times in the list)

We also know that $\displaystyle\sum_{k=1}^{N} a_k - \sum_{k=1}^{N} b_k = m + n\tau$ for some $m, n \in \mathbb{Z}$

Replace (say) a_N by $(\overline{a}_N =)\ a_N - m - n\tau$ so that

$$\sum_{k=1}^{N} a_k = \sum_{k=1}^{N} b_k$$

Let $g(z) = \prod_{k=1}^{N} \frac{\theta(z-a_k)}{\theta(z-b_k)}$

Then $g(z)$ is also ~~a~~ an elliptic function. Moreover

$\frac{f(z)}{g(z)}$ is elliptic and holomorphic, hence a constant

$$\Rightarrow \quad f(z) = C \cdot \prod_{k=1}^{N} \frac{\theta(z-a_k)}{\theta(z-b_k)} \quad (C \in \mathbb{C} \text{ is a constant})$$

$\square$

(22.2) Jacobi's imaginary transform.

Let $\tau' = \frac{-1}{\tau} \left(= -\frac{\bar{\tau}}{|\tau|^2} \right)$ so that $\operatorname{Im}(\tau')$ is still positive. Jacobi's imaginary transform is the name given to a relation between $\theta(z;\tau)$ and $\theta(z;\tau')$ functions.

This relation is best written for θ_3 instead of θ_1

Recall that (from HW 11, problem 2) $\theta_3(z;\tau)$ has the following properties.

- $$\theta_3(z;\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} e^{2\pi i n z}$$

$$= \prod_{n\geq 1} 1-q^{2n} \prod_{n\geq 0} 1+q^{2n+1} e^{2\pi i z} \prod_{n\geq 1} 1+q^{2n-1} e^{-2\pi i z}$$

- $\theta_3(z+1;\tau) = \theta_3(z;\tau)$

  $\theta_3(z+\tau;\tau) = q^{-1} e^{-2\pi i z}\, \theta_3(z;\tau)$

- $\theta_3(-z;\tau) = \theta_3(z;\tau)$

- $\theta_3(z;\tau)$ has zeroes (with mult. 1) at $\frac{1+\tau}{2} + \Lambda$.

  $\Lambda = \{m+n\tau \mid m,n\in\mathbb{Z}\}$

Consider the function $f(z) = e^{\pi i \tau' z^2}\, \theta_3\left(\frac{z}{\tau};\tau'\right)$

(1) $f(z)=0 \iff \theta_3\left(\frac{z}{\tau};\tau'\right) = 0$

$$\equiv \quad \frac{z}{\tau} = \frac{1}{2} + \frac{\tau'}{2} + m + n\tau' \quad \text{for some } m,n\in\mathbb{Z}$$

$$\equiv \quad z = \frac{\tau}{2} + \frac{\tau\tau'}{2} + m\tau + n\tau'\tau$$

$$\equiv \quad z = \frac{\tau}{2} - \frac{1}{2} + m\tau - n \qquad (\text{recall } \tau\tau' = -1)$$

$$\equiv \quad z \in \frac{1+\tau}{2} + \Lambda$$

(each with multiplicity 1)

(2) $$f(z+1) = e^{\pi i \tau'(z+1)^2}\, \theta_3\left(\frac{z+1}{\tau} ; \tau'\right)$$

$$= e^{\pi i \tau' z^2} \cdot e^{2\pi i \tau' z} \cdot e^{\pi i \tau'} \cdot \theta_3\left(\frac{z}{\tau} - \tau' ; \tau'\right)$$

$$\left(\text{since } \frac{1}{\tau} = -\tau'\right)$$

but $\theta_3(x+\tau' ; \tau') = e^{-\pi i \tau'}\, e^{-2\pi i x}\, \theta_3(x ; \tau')$

set $x = \frac{z}{\tau} - \tau'$. We get

$$\theta_3\left(\frac{z}{\tau} ; \tau'\right) = e^{-\pi i \tau'}\, e^{-2\pi i\left(\frac{z}{\tau} - \tau'\right)}\, \theta_3\left(\frac{z}{\tau} - \tau' ; \tau'\right)$$

$$\Rightarrow f(z+1) = e^{\pi i \tau' z^2} \cdot \underbrace{e^{2\pi i \tau' z} \cdot e^{\pi i \tau'} \cdot e^{\pi i \tau'}\, e^{2\pi i\left(\frac{z}{\tau} - \tau'\right)}}_{= 1 \text{ since } \frac{1}{\tau} = -\tau'} \cdot \theta_3\left(\frac{z}{\tau} ; \tau'\right)$$

$$= e^{\pi i \tau' z^2} \cdot \theta_3\left(\frac{z}{\tau} ; \tau'\right) = f(z)$$

(3) $$f(z+\tau) = e^{\pi i \tau'(z+\tau)^2}\, \theta_3\left(\frac{z+\tau}{\tau} ; \tau'\right)$$

$$= e^{\pi i \tau' z^2}\, e^{2\pi i \tau' z \tau}\, e^{\pi i \tau' \tau^2}\, \theta_3\left(\frac{z}{\tau} + 1 ; \tau'\right)$$

$$= e^{-\pi i \tau}\, e^{-2\pi i z}\, e^{\pi i \tau' z^2}\, \theta_3\left(\frac{z}{\tau} ; \tau'\right) \quad \left(\text{again since } \tau\tau' = -1\right)$$

$$= e^{-\pi i \tau}\, e^{-2\pi i z}\, f(z)$$

(22.3) In conclusion $f(z) = e^{\pi i \tau' z^2} \theta_3\left(\frac{z}{\tau}; \tau'\right)$

has exact same properties of $\theta_3(z;\tau)$. That is,

$\dfrac{\theta_3(z;\tau)}{f(z)}$ is holomorphic and elliptic and hence a

constant, say A.

$$\theta_3(z;\tau) = A\, e^{\pi i \tau' z^2}\, \theta_3\left(\frac{z}{\tau}; \tau'\right)$$

The value of the constant A can be computed using

$$\theta_1'(0;\tau) = -\pi i\, \theta_2(0;\tau)\, \theta_3(0;\tau)\, \theta_4(0;\tau)$$

and ends up being $A = -i\sqrt{i\tau'}$.

(22.4) From θ to Γ (degeneration)

Idea (for motivational purposes only!) . Recall

$$\theta(z;\tau) = G\, \theta_+(z;\tau)\, \theta_-(z;\tau)$$

?

as $\mathrm{Im}(\tau) \to \infty$

$$1 - e^{2\pi i z} = -e^{\pi i z}\left(\frac{e^{\pi i z} - e^{-\pi i z}}{2i}\right) \cdot 2i$$

$$= -2i\, e^{\pi i z}\, \frac{\sin(\pi z)}{\pi} \cdot \pi = -2\pi i\, e^{\pi i z}\, \frac{1}{\Gamma(z)\Gamma(1-z)}$$

To make this idea work we modify $\theta_+(z;\tau)$ as follows.

q - Gamma function is the name for this modified function.

$$\Gamma_q(x) = (\text{constant}) \cdot \frac{(1-q^2)^{-x}}{\theta_+(x\tau;\tau)}$$

$$= \prod_{n\geq 1}(1-q^{2n}) = G$$

Check $\Gamma_q(x+1) = \dfrac{1-q^{2x}}{1-q^2}\,\Gamma_q(x)$

$\dfrac{1-q^{2x}}{1-q^2}$ is sometimes denoted by $[x]_q$ "Gaussian numbers"

$$\lim_{q\to 1} [x]_q = x$$

So in the limit $q \to 1$, $\Gamma_q(x) \longrightarrow \Gamma(x)$

(Jackson 1905)