Lecture 3

(3.0) Recall that we introduced the notion of piecewise smooth path $\gamma: [a,b] \to \mathbb{C}$. (will be called just path or curve from now on).

Definition. Assume γ is smooth path in $D \subset \mathbb{C}$ open and let $f: D \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function (or more generally a continuous function).

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

If γ is only piecewise smooth, we take sum over subintervals where γ is smooth. That is, if

$[a,b] = [t_0 = a, t_1] \cup \cdots \cup [t_{n-1}, t_n = b]$ is subdivision for which γ is smooth for each $[t_0, t_1]$, $[t_1, t_2]$, …, $[t_{n-1}, t_n]$, then

$$\int_\gamma f(z)\,dz = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\,dt + \cdots + \int_{t_{n-1}}^{t_n} f(\gamma(t))\gamma'(t)\,dt$$

(3.1) $$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$ is nothing new again.

If $\gamma(t) = x(t) + i\,y(t)$ then $\gamma'(t) = x'(t) + i y'(t)$

and if $f(x+iy) = u(x,y) + i\,v(x,y)$, then after multiplying $f(z)$ with $\gamma'(t)$ we get

$$\int_\gamma f(z)\,dz = \int_a^b \left[u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t)\right]dt$$

$$+ \; i \int_a^b \left[u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\right]dt$$

and these are just definite integrals introduced in Calculus I and II.

e.g. $f(z) = 1$ constant function: $u(x,y) = 1$ and $v(x,y) = 0$.

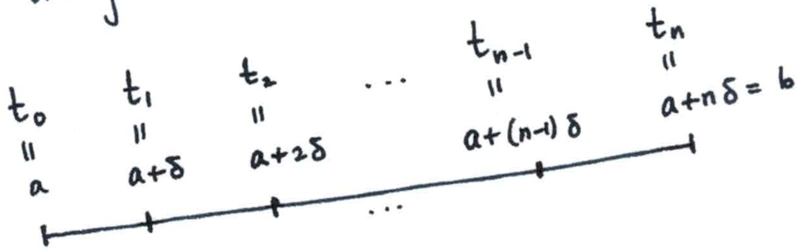
Then $\displaystyle\int_\gamma f(z)\,dz = \int_a^b x'(t)\,dt + i \int_a^b y'(t)\,dt$

$$= x(b) - x(a) + i\,(y(b) - y(a)) = \gamma(b) - \gamma(a).$$

(3.2) Riemann Sums: Let us recall the definition of definite integral from Calculus I; in our setting.

- For each $n \geq 1$ integer, subdivide $[a,b]$ into n equal pieces

$\delta = \dfrac{b-a}{n}$ and



- Define the sum

$$S_n := f(\gamma(t_0))\left(\gamma(t_1)-\gamma(t_0)\right) + f(\gamma(t_1))\left(\gamma(t_2)-\gamma(t_1)\right) + \dots$$
$$\dots + f(\gamma(t_{n-1}))\left(\gamma(t_n)-\gamma(t_{n-1})\right)$$

- Then $\displaystyle\int_\gamma f(z)\,dz = \lim_{n\to\infty} S_n.$

(3.3) Recall from Calculus III, length of a curve is given by

$$L = \int_a^b |\gamma'(t)|\, dt$$

Properties of integral:

(i) for $a, b \in \mathbb{C}$; $f(z), g(z)$ $\mathbb{C}$-differentiable functions and γ a path, we have

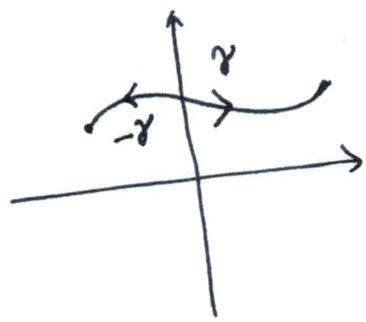
$$\int_\gamma \big(a f(z) + b g(z)\big)\, dz = a \int_\gamma f(z)\, dz + b \int_\gamma g(z)\, dz$$

(ii) Let $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ be two paths. Let $\gamma : [a,c] \to \mathbb{C}$ be the path given by joining γ_1 and γ_2.

Then $$\int_\gamma f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz$$

(iii) Let γ be a path in $\mathbb{C}$; $\gamma : [a,b] \to \mathbb{C}$. Define $-\gamma$ to be the opposite path: that is $-\gamma : [-b,-a] \to \mathbb{C}$ defined by

$$(-\gamma)(-t) = \gamma(t)$$



$$\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$$

(iv) Very important inequality :

$\gamma : [a,b] \to \mathbb{C}$ (path) $\qquad f(z)$: $\mathbb{C}$-differentiable function

- assume we have $M > 0$ such that $|f(\gamma(t))| \leq M$ for every $t \in [a,b]$. (This is always true as we will see later)
- let $L = \int_a^b |\gamma'(t)| \, dt$ = length of γ.

Then $$\left| \int_\gamma f(z) \, dz \right| \leq ML$$

Proof : use Riemann's sums as in section (3.2) :

$$|S_n| = \left| f(\gamma(t_0))(\gamma(t_1)-\gamma(t_0)) + \dots + f(\gamma(t_{n-1}))(\gamma(t_n)-\gamma(t_{n-1})) \right|$$

$$\leq |f(\gamma(t_0))| \, |\gamma(t_1)-\gamma(t_0)| + \dots + |f(\gamma(t_{n-1}))| \, |\gamma(t_n)-\gamma(t_{n-1})|$$

by triangle inequality

$$\leq M \left[|\gamma(t_1)-\gamma(t_0)| + |\gamma(t_2)-\gamma(t_1)| + \dots + |\gamma(t_n)-\gamma(t_{n-1})| \right]$$

since $|f(\gamma(t))| \leq M$

$$\leq ML$$

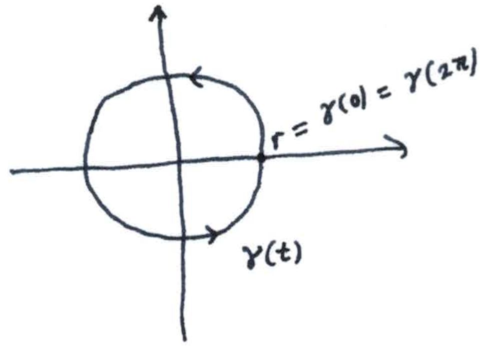
□

(3.4) One interesting example.

$$f(z) = \frac{1}{z} \qquad \gamma(t) = r(\cos t + i\sin(t)) \qquad 0 \le t \le 2\pi$$

(counterclockwise circle of radius r)

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i}$$



Proof. $\gamma'(t) = r(-\sin(t) + i\cos(t))$

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{r(\cos(t) + i\sin(t))}\, r(-\sin(t) + i\cos(t))\, dt$$

$$= \int_0^{2\pi} \frac{(-\sin(t) + i\cos(t))(\cos(t) - i\sin(t))}{(\cos(t) + i\sin(t))(\cos(t) - i\sin(t))}\,dt$$

$$= \int_0^{2\pi} \frac{[-\sin(t)\cos(t) + \cos(t)\sin(t)] + i[\sin^2(t) + \cos^2(t)]}{\cos^2(t) + \sin^2(t)}\,dt$$

$$= i\int_0^{2\pi} dt = 2\pi i \qquad \square$$

(3.5) Theorem. If $f: D \to \mathbb{C}$ is a $\mathbb{C}$-differentiable function such that there is $F: D \to \mathbb{C}$ with $F'(z) = f(z)$, then

$$\int_\gamma f(z)\,dz = 0 \quad \text{for every closed path } \gamma \text{ in } D.$$

($F(z)$ is called an anti-derivative of $f(z)$)

Proof.
$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$
$$= \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt$$
$$= \int_a^b \frac{d}{dt}\left(F(\gamma(t))\right)\,dt \quad \text{(by chain rule)}$$
$$= F(\gamma(b)) - F(\gamma(a)) \quad \text{(by fundamental theorem of calculus)}$$
$$= 0 \quad \text{since } \gamma(a) = \gamma(b) \qquad \square$$

Consequences of this theorem:

(i) If $f(z) = c_0 + c_1 z + \dots + c_n z^n$ is a polynomial, then

$$\int_\gamma f(z)\,dz = 0 \quad \text{for any closed path.}$$

Since $F(z) = c_0 z + \frac{c_1}{2} z^2 + \frac{c_2}{3} z^3 + \dots + \frac{c_n}{n+1} z^{n+1}$ is an anti-derivative of $f(z)$

(ii) $f(z) = \frac{1}{z}$ does not have an antiderivative in $\mathbb{C} \setminus \{0\}$

Proof. If it did, $\displaystyle\int_{\substack{\gamma = \text{circle of radius } r \\ \text{centered around } 0}} \frac{1}{z}\,dz$ would be 0 by Theorem (3.5)

but we already know it is $2\pi i$ by (3.4) $\square$

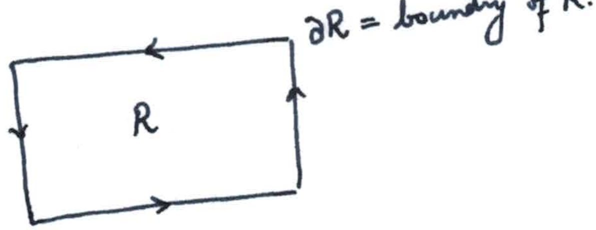
(3.6) Cauchy's Theorem. I.

[$D \subset \mathbb{C}$ an open set and $f : D \to \mathbb{C}$ a $\mathbb{C}$-differentiable function.]

Let R be a rectangle in D and γ = boundry of R. Then

$$\int_\gamma f(z)\,dz = 0$$

For the purposes of the proof, let us write ∂R = counterclockwise boundry of a rectangle R



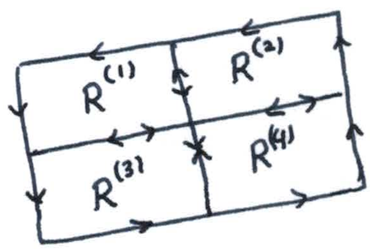
Let us write

$$I(R) = \int_{\partial R} f(z)\,dz$$

In order to prove that $I(R) = 0$, we will check that given $\varepsilon > 0$,

$$I(R) < \varepsilon \text{ (diagonal of } R) \text{ (perimeter of } R)$$

- Divide R into 4 equal pieces



Then $$\int_{\partial R} f(z)\,dz = \int_{\partial R^{(1)}} f(z)\,dz + \int_{\partial R^{(2)}} f(z)\,dz + \int_{\partial R^{(3)}} f(z)\,dz + \int_{\partial R^{(4)}} f(z)\,dz$$

i.e., $I(R) = I(R^{(1)}) + \dots + I(R^{(4)})$. By triangle inequality

$$|I(R)| \le |I(R^{(1)})| + |I(R^{(2)})| + |I(R^{(3)})| + |I(R^{(4)})|$$

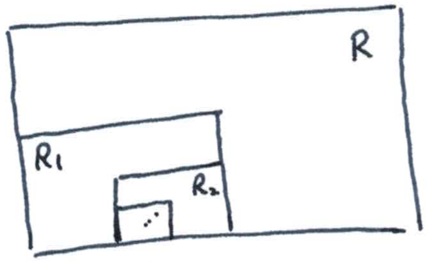
$$\le 4\,|I(R_1)|$$

where R_1 is one of $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$; for which $|I(R^{(i)})|$

is maximum.

Continue this process to get a sequence of rectangles

$$R \supset R_1 \supset R_2 \supset R_3 \supset \dots \supset R_n \supset \dots$$



① $|I(R)| \leq 4^n |I(R_n)|$

② $\begin{cases} \text{diagonal of } R_n = 2^{-n} \text{ diagonal of } R \\ \text{perimeter of } R_n = 2^{-n} \text{ perimeter of } R \end{cases}$

Pick w such that $w \in R_n$ for each n

With all this preparation, let us assume we are given $\varepsilon > 0$.

Choose $\delta > 0$ such that $|f(z) - f(w) - (z-w)f'(w)| < \varepsilon |z-w|$

for every z such that $|z-w| < \delta$

(this is just by definition of derivative).

Assume n is so large that R_n is contained in the disc

around w of radius δ. Then

$$I(R_n) = \int_{\partial R_n} f(z)\,dz = \int_{\partial R_n} (f(z) - f(w) - (z-w)f'(w))\,dz$$

(because $\int_{\partial R_n} (f(w) + (z+w)f'(w))\,dz = 0$ as the function under integral sign is just a polynomial in z)

Hence

$$|I(R_n)| \leq |f(z) - f(w) - (z-w) f'(w)| \cdot (\text{perimeter of } R_n) \quad [\text{by } (3.3)\,(\text{iv})]$$

$$< \varepsilon |z-w| \,(\text{perimeter of } R_n) \quad [\text{by our assumption on } n]$$

$$\leq \varepsilon \,(\text{diagonal of } R_n)(\text{perimeter of } R_n) \quad \left[\begin{array}{l}\text{for } z \text{ on } \partial R_n \\ w \text{ within } R_n \\ |z-w| \leq \text{length} \\ \text{of diag.}\end{array}\right]$$

[by ② of page 8]

$$= \varepsilon\, 4^{-n} (\text{diagonal of } R)(\text{perimeter of } R)$$

[by ① of page ⑧]

$$\Rightarrow \quad 4^{-n} |I(R)| \leq |I(R_n)| < 4^{-n} \varepsilon \,(\text{diagonal of } R)(\text{perimeter of } R)$$

$$\Rightarrow \quad |I(R)| < \varepsilon \,(\text{diagonal of } R)(\text{perimeter of } R) \qquad \square$$