Lecture 4

(4.0) Recall : we defined $\int_\gamma f(z)\,dz$, for a $\mathbb{C}$-differentiable function $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}$ is an open set and $\gamma : [a,b] \to D$ a path , as follows

- $$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$ if γ is smooth
- if γ is piecewise smooth, we take sum over the smooth parts of γ.

Cauchy's Theorem I. Let $R \subset D$ be a rectangle. Then

$$\int_{\partial R} f(z)\,dz = 0 \qquad (\partial R = \text{counterclockwise boundry of } R)$$

In this lecture we generalize this theorem to other closed curves γ.

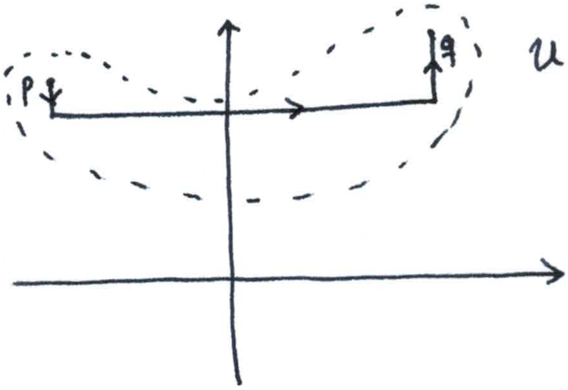
(4.1) Definition. A subset $\mathcal{U} \subset \mathbb{C}$ is said to be <u>(path)-connected</u> if for any two points $p, q \in \mathcal{U}$, there exists a path $\gamma : [a,b] \to \mathcal{U}$ such that $\gamma(a) = p$ and ~~$\gamma(q$~~ $\gamma(b) = q$.

e.g. $\mathbb{C} \setminus \{0\}$ is connected. $\mathbb{C} \setminus \mathbb{R}$ is not connected.

$\mathbb{R} \subset \mathbb{C}$ is connected.

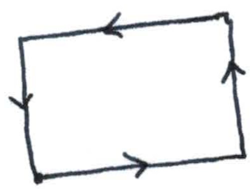
Fact : If $\mathcal{U} \subset \mathbb{C}$ is open and connected , then any two points of $\mathcal{U}$ can be connected by a "zig-zag path". That is, a path that consists entirely of horizontal or verticle line segments. (A proof of this fact is given at the end - optional reading).

Example of a zig-zag path :



(4.2) Morera's Theorem. Let $\mathcal{U} \subset \mathbb{C}$ be an open and connected subset. Let $f : \mathcal{U} \to \mathbb{C}$ be a continuous function such that for any rectangular closed path $\gamma : [a,b] \to \mathcal{U}$, we have

$$\int_\gamma f(z)\, dz = 0$$

$\left\{ \gamma = \right.$  in $\mathcal{U}$ $\left. \right\}$

Then there exists a $\mathbb{C}$ - differentiable function $F(z)$, $F : \mathcal{U} \to \mathbb{C}$ such that $F'(z) = f(z)$. (That is, $f(z)$ admits a $\mathbb{C}$ - differentiable anti derivative).

Proof of Morera's Theorem.

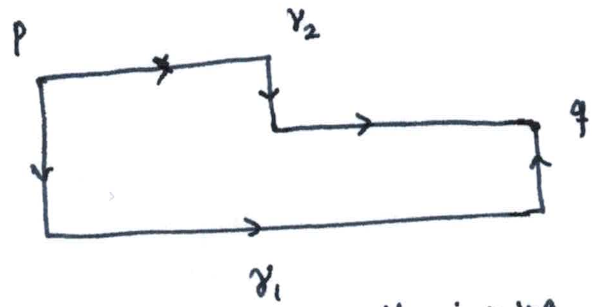
Fix a point $p \in \mathcal{U}$ and for any $w \in \mathcal{U}$, define

$$F(w) = \int_{\gamma} f(z)\, dz$$ where γ is a zig-zag path joining p and w.

Since $\int f(z)\, dz = 0$ over any closed rectangular path, the definition of $F(w)$ does not depend on the choice of γ

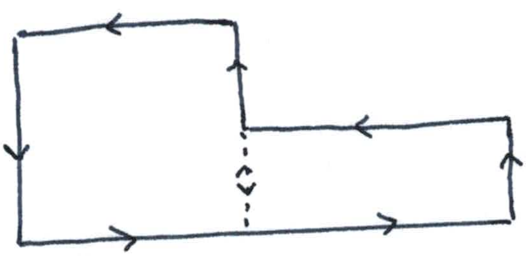
Ex.: Convince yourself of this statement.

e.g. if

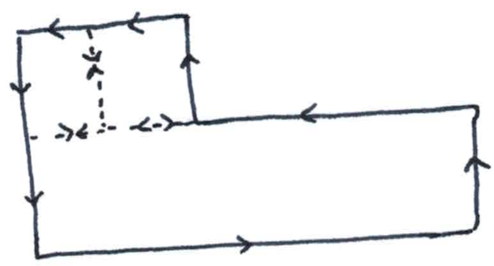


(γ_1, γ_2 are two zig-zag paths joining p and q)

$$\int_{\gamma_1} f(z) dz - \int_{\gamma_2} f(z)\, dz = \int_{\substack{\text{sum of rectangular}\\ \text{closed paths shown}\\ \text{below}}} f(z)\, dz = 0$$



or



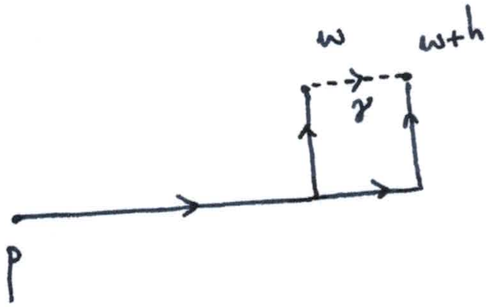
Take the one where dotted lines are still in $\mathcal{U}$.

Let us prove that F is $\mathbb{C}$-differentiable; and $F' = f$.

By Cauchy-Riemann equations, we need to check that

$$\frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y} = f \qquad \text{(see Theorem (1.4))}$$

Let us compute $\displaystyle \frac{\partial F}{\partial x}(w) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h}\left(F(w+h) - F(w)\right)$



$$= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h} \int_\gamma f(z)\,dz$$

where $\gamma : [0,1] \to \mathcal{U}$
$\gamma(t) = w + th$ is straight line joining w and $w+h$

$$= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{\cancel{h}} \int_0^1 f(w+th)\,\cancel{h}\,dt \qquad \text{(by definition)}$$

$$= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \int_0^1 f(w+th)\,dt$$

Claim. $\displaystyle \lim_{h \to 0} \int_0^1 f(w+th)\,dt = f(w)$

Proof. We need to prove that given $\varepsilon > 0$, we can find $\delta > 0$ such that for $|h| < \delta$ we have

$$\left| \int_0^1 f(w+th)\,dt - f(w) \right| < \varepsilon$$

Since f is continuous, pick $\delta > 0$ such that

$|k| < \delta$ implies $|f(w+k) - f(w)| < \varepsilon$. Now for this δ

we have, if $|h| < \delta$, then for every $t \in (0,1)$, $|th| < \delta$

and hence $\left| \int_0^1 f(w+th)\,dt - f(w) \right|$

$$= \left| \int_0^1 (f(w+th) - f(w))\,dt \right| < \varepsilon \cdot 1$$ $= $ length of $[0,1]$

as claimed. □

Hence $\frac{\partial F}{\partial x}(w) = f(w)$. Similar computation shows that

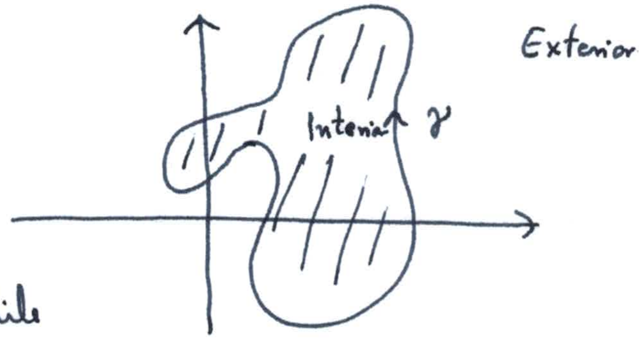
$\frac{\partial F}{\partial y}(w) = i\,f(w)$. Hence $\frac{\partial f}{\partial x} = -i\frac{\partial F}{\partial y} = f$ as desired. □

(4.3) Cauchy's Theorem (general version).

Let $D \subset \mathbb{C}$ be an open set and $f : D \to \mathbb{C}$ a $\mathbb{C}$-differentiable function. Let $\gamma : [a,b] \to D$ be <u>a simple closed path in D</u> <u>whose interior lies in D</u>. Then

$$\int_\gamma f(z)\,dz = 0$$

<u>Remark 1</u>. Every simple closed curve breaks the complex plane into two connected pieces $\to$ interior and exterior



* We assume (<u>always</u>) that γ is oriented <u>counterclockwise</u>

That is, interior is on the left, while travelling along γ.

<u>Remark 2</u>. We have already seen an example when interior of a simple closed curve is not within the domain of the function and the conclusion $\int_\gamma f(z)\,dz = 0$ is false.

Namely, $D = \mathbb{C} \setminus \{0\}$, $f(z) = \frac{1}{z}$ and $\gamma = r(\cos t + i\sin(t))$
$=$ circle of radius r

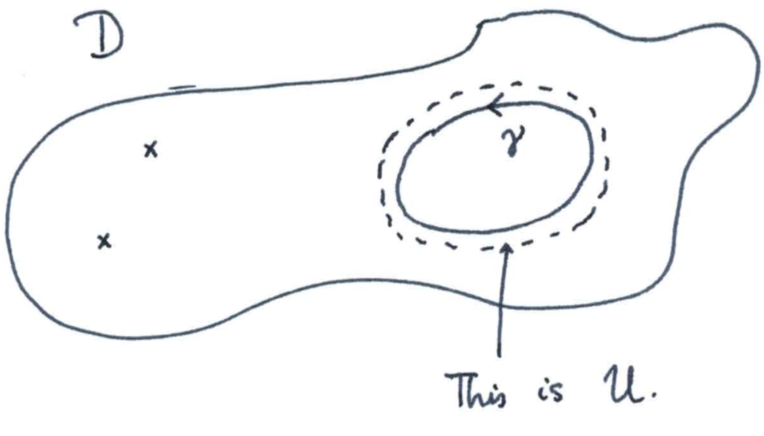
$0 \in$ Interior of γ but $0 \notin D$.

Proof of Cauchy's Theorem.

Step 1. Replace D by a smaller open set $\mathcal{U}$ containing γ such that

($\star$) for any simple closed curve μ inside $\mathcal{U}$, interior of μ is contained in $\mathcal{U}$.

e.g.



Chain of implications:

$\boxed{f \text{ is } \mathbb{C}\text{-differentiable}}$

$\Longrightarrow$ Cauchy's Theorem I

$$\boxed{\int_{\partial R} f(z)\,dz = 0 \quad \text{for any rectangle } R \subset D}$$

$\Big\Downarrow$ by property (☆) of $\mathcal{U}$

$$\boxed{\int_{\mu} f(z)\,dz = 0 \quad \text{for any rectangular path } \mu \text{ within } \mathcal{U} \subset D}$$

$\Longleftarrow$ Morera's Theorem

$\boxed{f \text{ admits an antiderivative on } \mathcal{U}}$

$\Big\Downarrow$ Theorem 3.5

$$\boxed{\int_{\gamma} f(z)\,dz = 0}$$