Lecture 5

(5.0) Recall that we have proved the following theorem

Cauchy's Theorem. Let $D \subset \mathbb{C}$ be an open set and $f : D \to \mathbb{C}$ a $\mathbb{C}$-differentiable function. Assume γ is a simple closed curve in D such that interior of γ is contained in D. Then $\displaystyle\int_\gamma f(z)\,dz = 0$.

Remark. (1) The conclusion $\displaystyle\int_\gamma f(z)\,dz = 0$ is false if either of the assumptions

- f is $\mathbb{C}$-differentiable
- Interior of $\gamma \subset D$

is dropped.

We have already seen an example when Interior $(\gamma) \not\subset D$ and the integral is non-zero. Let us see another example when f is not $\mathbb{C}$-differentiable:

Let $D = \mathbb{C}$, $f(z) = \bar{z}$, $\gamma = \cos(t) + i\sin(t)$ $\quad 0 \leq t \leq 2\pi$.

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} (\cos(t) - i\sin(t))(-\sin(t) + i\cos(t))\,dt$$

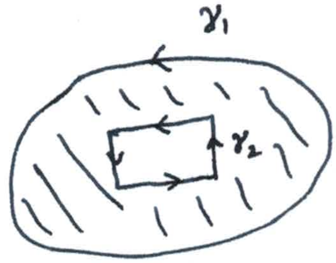
$$= \int_0^{2\pi} [-\cos(t)\sin(t) + \sin(t)\cos(t)] + i\,[\cos^2(t) + \sin^2(t)]\,dt$$

$$= 2\pi i$$

(2) As an application of Cauchy's Theorem, we can choose most convenient closed simple path for computations. That is, if γ_1 and γ_2 are two simple closed paths in D such that

(i) Interior $(\gamma_2) \subset$ Interior (γ_1)

(ii) Exterior $(\gamma_2) \cap$ Interior $(\gamma_1) \subset D$



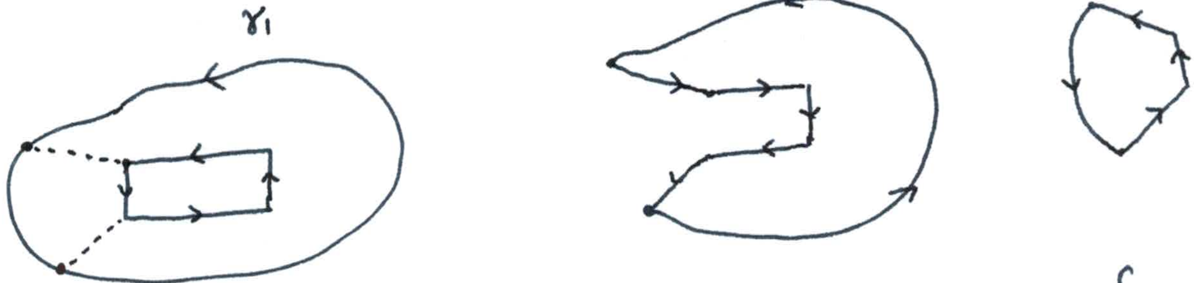
(The shaded part = Exterior $(\gamma_2) \cap$ Interior (γ_1) is assumed to be in D)

Then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

Proof: $\displaystyle\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$ can be written as sum of two integrals (or more) over simple closed curves satisfying hypothesis of Cauchy's Theorem, and hence has to be zero.

e.g.



$$\int_{\gamma_1} - \int_{\gamma_2} = \int_{\mu_1} + \int_{\mu_2}$$ $\square$

(5.1) In this lecture we prove the fundamental theorem of algebra. We begin by proving a slightly stronger version of Cauchy's theorem.

Strong version of Cauchy's theorem.

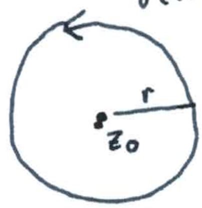
Assume $D \subset \mathbb{C}$ be an open set and $z_0 \in D$. Let $f : D \setminus \{z_0\} \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function such that

$$\lim_{z \to z_0} (z - z_0) f(z) = 0$$

Then $\oint_\gamma f(z)\,dz = 0$ for any simple closed path γ in $D \setminus \{z_0\}$ such that interior of γ is contained in D.

Proof. If z_0 is not in the interior of γ, then the statement follows from Cauchy's theorem. Let us assume z_0 is in the interior of γ. By remark (2) above we may replace γ by a circle around z_0 (say of radius r)

$$\gamma(t) = z_0 + r(\cos(t) + i\sin(t))$$
$$0 \le t \le 2\pi$$



Since $\lim_{z \to z_0} (z - z_0) f(z) = 0$, given $\varepsilon > 0$ we can choose $\delta > 0$ such that $|z - z_0| < \delta$ implies $|z - z_0|\,|f(z)| < \varepsilon$.

Take $0 < r < \delta$. Then

$$\left| \int_\gamma f(z)\,dz \right| < \frac{\varepsilon}{|z - z_0|} 2\pi r$$

$$= \frac{\varepsilon}{r} 2\pi r = 2\pi \varepsilon.$$

Hence $\int_\gamma f(z)\,dz = 0$ as claimed. $\square$

(5.2) Cauchy's integral formula.

Let $D \subset \mathbb{C}$ be an open set, γ a simple closed path in D whose interior lies in D. Let $f : D \to \mathbb{C}$ be $\mathbb{C}$-differentiable and let $a \in \text{Interior}(\gamma)$. Then

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$

Proof. Let $g(z) = \dfrac{f(z) - f(a)}{z-a}$ for $z \neq a$ defined on $D \setminus \{a\}$. Since f is continuous $\lim_{z \to a} (z-a)\, g(z) = 0$. By the strong version of Cauchy's theorem $\int_\gamma g(z)\, dz = 0$. That is

$$0 = \int_\gamma \frac{f(z)}{z-a}\, dz - \int_\gamma \frac{f(a)}{z-a}\, dz.$$

Now use the example computation given in Lecture 3, paragraph (3.4) and Remark (2) of (5.0) above to see that $\int_\gamma \frac{1}{z-a}\, dz = 2\pi i$. That is

$$\int_\gamma \frac{f(z)}{z-a}\, dz = f(a) \int_\gamma \frac{1}{z-a}\, dz = 2\pi i\, f(a)$$

□

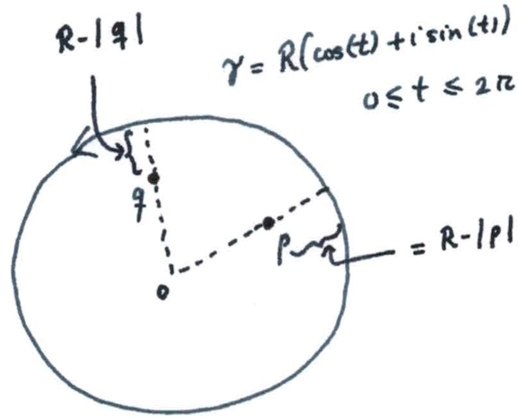
(5.3) Definition. A $\mathbb{C}$-differentiable function $\mathbb{C} \to \mathbb{C}$ (that is, domain $= \mathbb{C}$) is called <u>entire function</u>.

Liouville's Theorem. Let $f(z)$ be an entire function which is bounded, that is, there exists $M > 0$ (real) such that $|f(z)| < M$ for every $z \in \mathbb{C}$. Then f is constant.

Proof. Let $p, q \in \mathbb{C}$. We claim that $f(p) = f(q)$. To prove this pick $R > 0$ such that $|p|, |q| < R$ and let γ = circle of radius R. Then by Cauchy's integral formula

$$f(q) - f(p) = \frac{1}{2\pi i} \int_\gamma \left(\frac{f(z)}{z-q} - \frac{f(z)}{z-p} \right) dz$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{(q-p)\, f(z)}{(z-q)(z-p)}\, dz$$



Now for z on γ, $|z - q| > R - |q|$

$$\Rightarrow \left| \frac{(q-p)\, f(z)}{(z-q)(z-p)} \right| < \frac{|q-p|\, M}{(R-|q|)(R-|p|)}$$

Hence $$|f(q) - f(p)| < \frac{1}{2\pi} \frac{|q-p|\, M}{(R-|q|)(R-|p|)}\, 2\pi R$$

$$= |q-p|\, M \boxed{\frac{R}{(R-|q|)(R-|p|)}}$$

tends to 0 as $R \to \infty$.

That is, we can make $|f(q) - f(p)|$ as small as we want by picking R large enough.

$$\Rightarrow |f(q) - f(p)| = 0 \Rightarrow f(q) = f(p)$$

$\square$

(5.4) Polynomials and fundamental theorem of algebra.

Let $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a polynomial of degree $n \geq 1$, that is, $c_n \neq 0$.

Fundamental Theorem of Algebra. There exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof. Assume this is not true. That is, $p(w) \neq 0$ for every $w \in \mathbb{C}$.

Then $f(z) = \dfrac{1}{p(z)}$ is an entire function (i.e., defined and $\mathbb{C}$-differentiable everywhere). We claim that $f(z)$ is bounded as well. To see this write

$$p(z) = c_n z^n \left[1 + \frac{c_{n-1}}{c_n z} + \frac{c_{n-2}}{c_n z^2} + \cdots + \frac{c_0}{c_n z^n}\right]$$

Choose $R > 0$ large enough that $\left|\dfrac{c_k}{c_n R^{n-k}}\right| < \dfrac{1}{2n}$ for each $k = 0, \ldots, n-1$.

Then
$$\begin{aligned}
|p(z)| &= |c_n|\,|z|^n \left|1 + \frac{c_{n-1}}{c_n z} + \cdots + \frac{c_0}{c_n z^n}\right| \\
&\geq |c_n|\,|z|^n \left(1 - \left|\frac{c_{n-1}}{c_n z}\right| - \cdots - \left|\frac{c_0}{c_n z^n}\right|\right) \quad \left[\begin{array}{l}\text{by triangle inequality} \\ |a| - |b| \leq |a-b|\end{array}\right] \\
&\geq |c_n|\, R^n \left(1 - \frac{1}{2n} - \frac{1}{2n} \cdots - \frac{1}{2n}\right) \quad \text{for } |z| > R \\
&= \frac{|c_n|\, R^n}{2}
\end{aligned}$$

Hence for $|z| > R$, $\dfrac{1}{|p(z)|} \leq \dfrac{2}{|c_n|\, R^n}$ (bounded)

Moreover since $\dfrac{1}{p(z)}$ is continuous, it is bounded on $|z| \leq R$

Note: the previous statement has appeared in Calculus I and II: namely a continuous function on a closed and bounded domain attains absolute max. value. A proof is given later (optional reading).

Therefore $f(z) = \dfrac{1}{p(z)}$ is a bounded entire function. Hence, by Liouville's Theorem, it must be constant. This is a contradiction since degree of p was assumed to be greater than (or equal to) 1. $\square$

(5.5) As a consequence of FTA, every polynomial $p(z)$ can be written as product of linear (degree 1) polynomials. To see this, let $z_0 \in \mathbb{C}$ be such that $p(z_0) = 0$. Then

$$\begin{aligned} p(z) &= p(z) - p(z_0) \\ &= c_n (z^n - z_0^n) + c_{n-1}(z^{n-1} - z_0^{n-1}) + \cdots + c_1 (z - z_0) \end{aligned}$$

Now $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

Hence $p(z) = p(z) - p(z_0) = (z - z_0) \cdot$ polynomial of degree $n-1$

Continue this way to write

$$p(z) = (z-z_0)(z-z_1) \cdots (z - z_{n-1}) \cdot c \qquad (c = c_n)$$

Note: $z_0, z_1, \ldots, z_{n-1}$ are n-roots of $p(z)$ of degree n.

They <u>do not have to be distinct</u> ← careful.