Lecture 6

(6.0) Recall that we have proved the following results about $\int_\gamma f(z)dz$.

Cauchy's Theorem. $\int_\gamma f(z)\,dz = 0$ for γ a simple closed curve whose interior is contained in D and f is $\mathbb{C}$ differentiable on D.

Cauchy's Integral formula $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = f(a)$ with same assumptions as before and a is a point in the interior of γ.

Applications I. Fundamental Theorem of algebra

Application II. Liouville's Theorem (every bounded entire function is constant).

(6.1) Assume that $f : D \to \mathbb{C}$ is $\mathbb{C}$-differentiable such that for every $n \geq 1$, $f^{(n)} : D \to \mathbb{C}$ exists
(n^{th} derivative of f)

Remark: We will see later that this stronger hypothesis is not needed. In fact, it follows from the fact that f is $\mathbb{C}$-differentiable!

Then $$\boxed{\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{f^{(n)}(a)}{n!}}$$ for $n = 0, 1, 2, \dots$

(recall $n! = 1\,(2)\,(3)\cdots(n)$) with the convention that $0! = 1$.
(factorial)

Note: for $n = 0$, this is just Cauchy's Integral formula

Proof. (integration by parts)

For each $n \geq 1$, the function $\dfrac{f(z)}{(z-a)^{n+1}} - \dfrac{1}{n}\dfrac{f'(z)}{(z-a)^n}$ has an antiderivative, namely $-\dfrac{1}{n}\dfrac{f(z)}{(z-a)^n}$:

$$\frac{d}{dz}\left(-\frac{1}{n}\frac{f(z)}{(z-a)^n}\right) = -\frac{1}{n}(-n)(z-a)^{-n-1} f(z) - \frac{1}{n}\frac{f'(z)}{(z-a)^n}$$

$$= \frac{f(z)}{(z-a)^{n+1}} - \frac{1}{n}\frac{f'(z)}{(z-a)^n}$$

$$\Rightarrow \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{1}{n}\int_\gamma \frac{f'(z)}{(z-a)^n}\,dz = \frac{1}{n(n-1)}\int_\gamma \frac{f''(z)}{(z-a)^{n-1}}$$

$$= \cdots = \frac{1}{n!}\int_\gamma \frac{f^{(n)}(z)}{z-a}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

by Cauchy's integral formula

□

(6.2) Rational functions and limit at infinity.

Definition of limit at infinity. We say $\lim_{z \to \infty} f(z) = L$ if for every $\varepsilon > 0$, there exists $R > 0$ such that

$|z| > R$ implies $|f(z) - L| < \varepsilon$.

A rational function $f(z)$ is ratio of two polynomial functions

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + \dots + a_n z^n}{b_0 + b_1 z + \dots + b_m z^m} \qquad (a_n, b_m \neq 0)$$

Note: if $n = m$ then $\lim_{z \to \infty} f(z) = \frac{a_n}{b_n}$. if $m > n$ then this limit is zero.

The rational function $\frac{p(z)}{q(z)}$ is $\mathbb{C}$-differentiable on $\mathbb{C} \setminus \{z_0 \mid q(z_0) = 0\}$.

Namely, if $q(z) = (z - z_1) \dots (z - z_m) b_m$; then $\frac{p(z)}{q(z)}$ is $\mathbb{C}$-diff. on $\mathbb{C} \setminus \{z_1, \dots, z_m\}$ (Careful: $z_1 \dots z_m$ need not be distinct).

For simplicity, let $b_m = 1$. Let γ be a simple closed curve such that $z_1, z_2, \dots, z_m$ are in the interior of γ

- If $m \geq n + 2$, then
$$\int_\gamma \frac{p(z)}{q(z)} dz = 0$$

- If $m = n+1$, then $\frac{1}{2\pi i} \int_\gamma \frac{p(z)}{q(z)} dz = a_n$

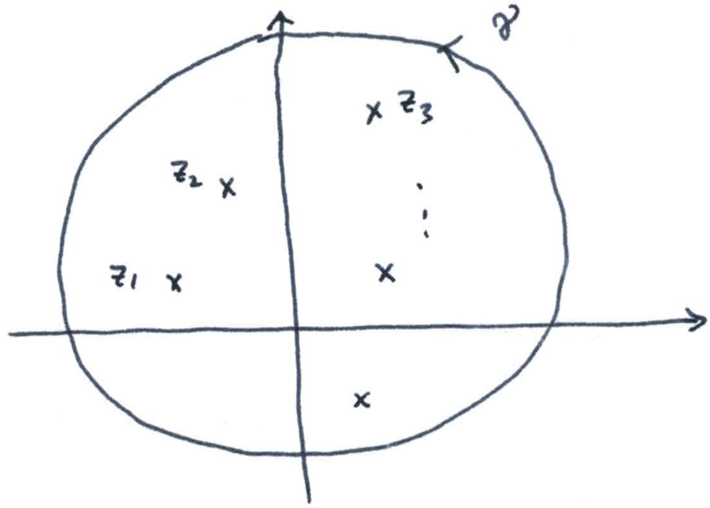
Note: if $m \leq n$ then (by dividing polynomials)

$$\frac{p(z)}{q(z)} = \underset{\text{polynomial}}{h(z)} + \frac{p_1(z)}{q(z)} \quad \text{and} \quad \int_\gamma h(z)\, dz = 0$$

$p_1(z)$ of degree $< m$

by Cauchy's Theorem or Theorem (3.5).

(6.3) Proof of $\displaystyle\int_\gamma \frac{p(z)}{q(z)}\,dz = 0$ if (degree of q) $\geq$ (degree of p) + 2



roots of q

By remark (2) of (5.0)

$$\int_\gamma f(z)\,dz = \int_{C_R} f(z)\,dz$$

C_R = (counterclockwise) circle of radius R, centered at 0.

for R large enough ($R > |z_1|, |z_2|, \dots$)

and it does not depend on R. For z such that $|z| = R$,

$$|p(z)| = |a_0 + a_1 z + \dots + a_n z^n| \leq |a_0| + |a_1| R + \dots + |a_n| R^n$$

$$< (|a_0| + |a_1| + \dots + |a_n|) R^n \quad \text{(assuming } R > 1)$$

$$|q(z)| = |b_0 + b_1 z + \dots + b_m z^m| \qquad (b_m = 1)$$

$$= R^m \left| 1 + \frac{b_{m-1}}{z} + \dots + \frac{b_1}{z^{m-1}} + \frac{b_0}{z^m} \right|$$

$$\geq R^m \left(1 - \frac{|b_{m-1}|}{R} - \frac{|b_{m-2}|}{R^2} - \dots - \frac{|b_0|}{R^m} \right)$$

Take R large enough so that each $\dfrac{|b_{m-k}|}{R^k} < \dfrac{1}{2m}$

$$|q(z)| > \frac{1}{2} R^m$$ [see the proof of FTA in (5.4)]

Then on C_R, $\left|\frac{p(z)}{q(z)}\right| < 2\left(|a_0| + \dots + |a_n|\right)\frac{R^n}{R^m}$

Hence
$$\left|\int_{C_R}\frac{p(z)}{q(z)}\,dz\right| < 2\left(|a_0| + \dots + |a_n|\right)\frac{R^n}{R^m}\,2\pi R$$
$$= \frac{4\pi\left(|a_0| + \dots + |a_n|\right)}{R^{m-n-1}}$$

If $m \geq n+2$, we can make $\frac{1}{R^{m-n-1}}$ as small as we want. That is,

$\lim_{R\to\infty}\frac{1}{R^{m-n-1}} = 0$. Hence $\int_{C_R}\frac{p(z)}{q(z)}\,dz = 0$.

(6.4) Proof of $\frac{1}{2\pi i}\int_{\gamma}\frac{p(z)}{q(z)}\,dz = a_n$ if $m = n+1$

(recall, $m =$ degree of q
$n =$ degree of p)

Again replace γ by a C_R = circle of radius R centered at 0.

By example (3.4) or Cauchy's formula $a_n = \frac{1}{2\pi i}\int_{C_R}\frac{a_n}{z}\,dz$

Hence
$$\frac{1}{2\pi i}\int_{C_R}\frac{p(z)}{q(z)}\,dz - a_n = \frac{1}{2\pi i}\int_{C_R}\left(\frac{p(z)}{q(z)} - \frac{a_n}{z}\right)dz$$
$$= \frac{1}{2\pi i}\int_{C_R}\frac{z\,p(z) - a_n\,q(z)}{z\,q(z)}\,dz$$

Now $\text{degree}\,(z\,p(z) - a_n\, q(z)) \le n$

$\text{degree}\,(z\, q(z)) = n+2$ and by previous part (6.3).

$$\frac{1}{2\pi i} \int_{C_R} \frac{z\,p(z) - a_n\, q(z)}{z\, q(z)}\, dz = 0.$$

(6.5) Partial fractions. (from Calculus II)

- $p(z) = a_0 + a_1 z + \dots + a_n z^n$ $(a_n \neq 0)$ polynomial of degree n.
- $q(z) = (z - z_1) \dots (z - z_m)$

  assume $m \ge n+1$ and $z_1, \dots, z_m$ are distinct.

Then

$$\frac{p(z)}{q(z)} = \left[\frac{p(z_1)}{(z_1 - z_2)\dots(z_1 - z_m)}\right]\frac{1}{z - z_1} + \left[\frac{p(z_2)}{(z_2 - z_1)(z_2 - z_3)\dots(z_2 - z_m)}\right]\frac{1}{z - z_2}$$

$$+ \dots + \left[\frac{p(z_k)}{(z_k - z_1)\dots(z_k - z_{k-1})(z_k - z_{k+1})\dots(z_k - z_m)}\right]\frac{1}{z - z_k} +$$

$$\dots + \left[\frac{p(z_m)}{(z_m - z_1)\dots(z_m - z_{m-1})}\right]\frac{1}{z - z_m}$$

If $q(z)$ has <u>multiple roots</u>, that is, $q(z) = (z-a)^\ell q_1(z)$ $(\ell \ge 2)$

then in partial fractions the term $\dfrac{A_1}{z-a} + \dfrac{A_2}{(z-a)^2} + \dots + \dfrac{A_\ell}{(z-a)^\ell}$

appears. The numbers $A_1, \dots, A_\ell$ can be computed using Cauchy's formula e.g. Consider $f(z) = \dfrac{z+1}{(z-1)^2(z-2)}$

$$= \frac{A_1}{z-1} + \frac{A_2}{(z-1)^2} + \frac{A_3}{z-2}$$

. To compute A_3, integrate both sides along a small circle around 2.

$$A_3 = \frac{1}{2\pi i}\int_{C_2} \frac{A_3}{z-2}\,dz = \frac{1}{2\pi i}\int_{C_2} f(z)\,dz = \frac{2+1}{(2-1)^2} = 3$$

Cauchy's formula

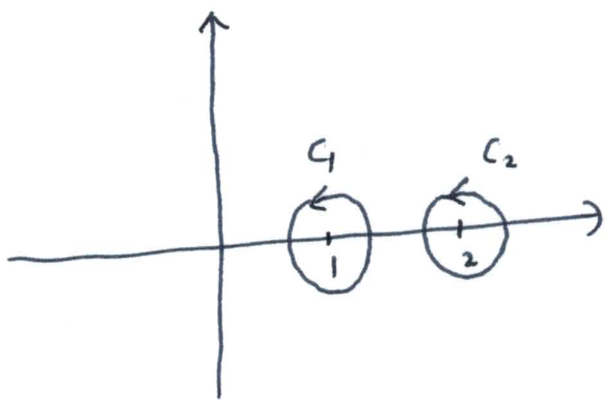
Similarly, $A_1 = \dfrac{1}{2\pi i}\displaystyle\int_{C_1} f(z)\,dz = \dfrac{1}{2\pi i}\int_{C_1}\left(\dfrac{z+1}{z-2}\right)\dfrac{1}{(z-1)^2}\,dz$

$$= \frac{d}{dz}\left(\frac{z+1}{z-2}\right)\bigg|_{z=1} = \left[\frac{1}{z-2} - \frac{z+1}{(z-2)^2}\right]_{z=1} = -1-2=-3$$

$$A_2 = \frac{1}{2\pi i}\int_{C_1}(z-1)f(z)\,dz = \frac{1}{2\pi i}\int_{C_1}\left(\frac{z+1}{z-2}\right)\frac{1}{z-1}\,dz = \frac{2}{-1} = -2$$

So $f(z) = \dfrac{-3}{z-1} + \dfrac{-2}{(z-1)^2} + \dfrac{3}{z-2}$ (verify this).

Simple closed curves C_1 and C_2:



(6.6) Every polynomial over $\mathbb{R}$ can be factored into linear or quadratic polynomials over $\mathbb{R}$

(was perhaps mentioned in Calculus II. Now we will see a proof).

Let $p(x) = a_0 + a_1 x + \dots + a_n x^n$ be a polynomial with $a_0, \dots, a_n \in \mathbb{R}$.

Then by FTA $p(x) = a_n(x - z_1) \dots (x - z_n)$ where $z_1 \dots z_n \in \mathbb{C}$.

Note: if $z \in \mathbb{C}$ is a root, $a_0 + a_1 z + \dots + a_n z^n = 0$, then taking conjugate of this equation ($a_i \in \mathbb{R}$, so $\bar{a}_i = a_i$) we get

$$a_0 + a_1 \bar{z} + a_2 (\bar{z})^2 + \dots + a_n (\bar{z})^n = 0 \Rightarrow \bar{z} \text{ is also a root.}$$

So, the (complex) roots of $p(x)$ are either real or come in pairs

$(\alpha, \bar{\alpha})$ $\alpha \in \mathbb{C}$.

Real roots $\longrightarrow$ linear factors of $p(x)$

$(\alpha, \bar{\alpha})$ pair $\longrightarrow$ $(x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha}$

$= x^2 - 2\,\mathrm{Re}(\alpha)\, x + |\alpha|^2$ quadratic poly. over $\mathbb{R}$.

$\square$