Lecture 7

(7.0) Recall Cauchy's formula. Let $D \subset \mathbb{C}$ be a non-empty open set and $f: D \to \mathbb{C}$ a $\mathbb{C}$-differentiable function. Then for every $a \in D$

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a}\, dz$$

where γ is $\overset{\text{a}}{\vee}$ counterclockwise circle centered at a, of radius small enough so that γ and its interior are contained in D.

Theorem. Assuming the hypotheses on D, f, a, γ etc. written above, we have

$$f'(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^2}\, dz.$$ Furthermore $f'(z)$ is $\mathbb{C}$-differentiable on D and

$$f''(a) = \frac{2}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^3}\, dz.$$

Meaning, if f is $\mathbb{C}$-differentiable ~~on~~ then f is differentiable to any order and (by calculation given in class - Lecture 6, section 6.1)

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

To emphasize this nice property, the term <u>$\mathbb{C}$-differentiable</u> will be replaced by <u>holomorphic</u> (people also use <u>analytic</u>).

Proof. Let us begin by proving that $f'(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^2}\, dz.$

By definition $\displaystyle f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$$= \lim_{h\to 0} \frac{1}{h\cdot 2\pi i}\int_\gamma \frac{f(z)}{z-a-h} - \frac{f(z)}{z-a}\,dz \quad (\text{by Cauchy's formula})^*$$

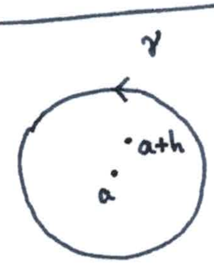
$$= \frac{1}{2\pi i}\lim_{h\to 0}\frac{1}{h}\int_\gamma \frac{f(z)\big((z-a)-(z-a-h)\big)}{(z-a)(z-a-h)}\,dz = \frac{1}{2\pi i}\lim_{h\to 0}\int_\gamma \frac{f(z)}{(z-a)(z-a-h)}\,dz$$

We need to show that $\displaystyle \lim_{h\to 0}\int_\gamma \frac{f(z)}{(z-a)(z-a-h)}\,dz = \int_\gamma \frac{f(z)}{(z-a)^2}\,dz.$

By definition of limit, this means:

for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |h| < \delta \implies \left|\int_\gamma \frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2}\,dz\right| < \varepsilon$$



* since $h \to 0$, we are assuming that it is small enough: $|h| < r$

Now

$$\left|\int_\gamma \left(\frac{f(z)}{(z-a-h)(z-a)} - \frac{f(z)}{(z-a)^2}\right)dz\right| = \left|\int_\gamma \frac{f(z)\cdot h}{(z-a-h)(z-a)^2}\,dz\right|$$

$$\leq \frac{M\cdot|h|}{(r-|h|)\,r^2}\,2\pi r$$

where
- $M \in \mathbb{R}$ is such that $|f(z)| \leq M$ for every z on γ.
- $|z-a| = r$
- $|z-a-h| \geq |z-a| - |h| = r - |h|$
- $2\pi r$ = length of γ.

To make this number smaller than ε

pick $\displaystyle \delta \leq \min\left(\frac{r}{2}, \varepsilon\frac{r^2}{4\pi M}\right)$

Then
$$\left| \int_\gamma \frac{f(z)}{(z-a-h)(z-a)} - \frac{f(z)}{(z-a)^2}\, dz \right| \le \frac{M\cdot|h|}{(r-|h|)\,r}\, 2\pi$$

$$\le \frac{M\,|h|}{\frac{r}{2}\cdot r}\cdot 2\pi = \frac{4\pi M}{r^2}|h| \qquad \left(\text{for } |h| < \delta,\ r-|h| > \frac{r}{2} \right)$$

$$\delta = \min\left(\frac{r}{2},\ \varepsilon\frac{r^2}{4\pi M}\right)$$

$< \varepsilon$ as desired.

Now we prove that f' is again $\mathbb{C}$-differentiable. That is, for every $a \in D$, $\lim\limits_{h\to 0} \dfrac{f'(a+h) - f'(a)}{h}$ exists and is equal to $\dfrac{2}{2\pi i}\displaystyle\int_\gamma \frac{f(z)}{(z-a)^3}\, dz$. The proof is along the same lines as the one given above. We begin by writing (for $|h| < r$)

$$\frac{1}{h}\left(f'(a+h) - f'(a)\right) = \frac{1}{h\cdot 2\pi i}\int_\gamma \left(\frac{f(z)}{(z-a-h)^2} - \frac{f(z)}{(z-a)^2} \right) dz$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(z)\left((z-a)^2 - (z-a-h)^2\right)}{h\,(z-a-h)^2(z-a)^2}\, dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\left(2(z-a)-h\right)}{(z-a-h)^2(z-a)^2}\, dz$$

$$= \frac{2}{2\pi i}\int_\gamma \frac{f(z)}{(z-a-h)^2(z-a)}\, dz - \frac{h}{2\pi i}\int_\gamma \frac{f(z)}{(z-a-h)^2(z-a)^2}\, dz$$

In the limit $h \to 0$, the second term goes to 0 and we are left with

$$\lim_{h\to 0}\int_\gamma \frac{f(z)}{(z-a-h)^2(z-a)}\, dz = \int_\gamma \frac{f(z)}{(z-a)^3}\, dz \qquad \text{to prove.}$$

Meaning, given $\varepsilon > 0$, we have to find $\delta > 0$ such that $0 < |h| < \delta$ implies

$$\left| \int_\gamma \left(\frac{f(z)}{(z-a-h)^2 (z-a)} - \frac{f(z)}{(z-a)^3} \right) dz \right| < \varepsilon .$$

The left-hand side is simplified as

$$\text{L.H.S.} = \left| \int_\gamma \frac{f(z)\left(2h(z-a) - h^2\right)}{(z-a-h)^2 (z-a)^3} dz \right|$$

Let $M \in \mathbb{R}$ be such that $|f(z)| \leq M$ for every z on γ.

$$|2h(z-a) - h^2| = |h| \left| (2(z-a) - h) \right| \leq |h| \left(2|z-a| + |h| \right) \leq 3r|h|$$

(since $|z-a| = r$ and $|h| < r$)

Also, assuming $|h| < \frac{r}{2}$, $|z-a-h| > \frac{r}{2}$

Therefore,

$$\left| \int_\gamma \frac{f(z)\left(2h(z-a) - h^2\right)}{(z-a-h)^2 (z-a)^3} dz \right| \leq \frac{M \cdot 3r|h|}{\left(\frac{r}{2}\right)^2 r^3} 2\pi r = \frac{24\pi M |h|}{r^3}$$

Take $\delta = \min\left(\frac{r}{2}, \varepsilon \frac{r^3}{24\pi M} \right)$. Then for $|h| < \delta$ we get

$$\left| \int_\gamma \frac{f(z)}{(z-a-h)^2 (z-a)} dz - \int_\gamma \frac{f(z)}{(z-a)^3} dz \right| \leq \frac{24\pi M|h|}{r^3} < \varepsilon .$$

$\square$

From now onwards we use the word <u>holomorphic</u> instead of <u>$\mathbb{C}$-differentiable</u>.

(7.1) Convergence and uniform convergence : sequences.

Definition : A sequence of complex numbers $\{s_1, s_2, s_3, \dots\}$ is said to converge to $s \in \mathbb{C}$ (written as $\lim_{n\to\infty} s_n = s$ or $s_n \to s$ as $n \to \infty$) if for every $\varepsilon > 0$, there exists $N > 0$ (integer) such that $|s_n - s| < \varepsilon$ for every $n \geq N$.

Cauchy's criterion (proof will be given in Optional Reading 2).

A sequence of complex numbers $\{s_1, s_2, \dots\}$ converges if and only if for every $\varepsilon > 0$, there is $N > 0$ such that $|s_n - s_m| < \varepsilon$ for every $n, m \geq N$.

Remark. Cauchy's criterion has the advantage that the actual limit s $(= \lim_{n\to\infty} s_n)$ does not have to be known.

Now let $D \subset \mathbb{C}$ be an open set and $f_n : D \to \mathbb{C}$ $(n = 1, 2, 3, \dots)$ be a sequence of functions.

Pointwise Convergence : $\lim_{n\to\infty} f_n = f$ pointwise, if for every $z \in D$ $\lim_{n\to\infty} f_n(z) = f(z)$. Namely, given $z \in D$ and $\varepsilon > 0$, we can find N (depending on z and ε) such that $|f_n(z) - f(z)| < \varepsilon$ for every $n \geq N$. (Or, by Cauchy's criterion $|f_n(z) - f_m(z)| < \varepsilon$ for every $n, m \geq N$.)

Uniform convergence: $\lim_{n\to\infty} f_n = f$ uniformly on a subset $A \subset D$ if for every $\varepsilon > 0$ we can find $N > 0$ (depending on ε and A, but not on a specific $z \in A$) such that

$$|f_n(z) - f(z)| < \varepsilon \quad \text{for every } n \geq N \text{ and } z \in A.$$

(Or by Cauchy's criterion $|f_n(z) - f_m(z)| < \varepsilon$ for every $n, m \geq N$ and $z \in A$)

Meaning that $N > 0$ can be chosen at once for every $z \in A$ (uniformly).

(7.2) For us the relevant subsets $A \subset D$ will be the ones called <u>compact sets</u>. These were defined in Optional Reading 1 as closed and bounded:

$$\text{Compact} = \text{Closed and Bounded}.$$

- $A \subset \mathbb{C}$ is closed if for every convergent sequence $\lim_{n\to\infty} z_n = z$

  $$z_n \in A \text{ for every } n \Rightarrow z \in A.$$

- $A \subset \mathbb{C}$ is bounded if there exists $R > 0$ such that

  $$|z| < R \quad \text{for every } z \in A.$$

From now onwards $\{f_n\}$ converges to f uniformly, written as $\lim_{n\to\infty} f_n = f$ uniformly; means <u>uniformly on compact subsets contained in D</u>.

(7.3) The main point of uniform convergence is the following theorem.

Theorem. Let $D \subset \mathbb{C}$ be an open set and $f_n : D \to \mathbb{C}$ a sequence of functions converging uniformly to $f : D \to \mathbb{C}$.

(1) If f_n is continuous for every n, then f is continuous.

(2) Let $\gamma : [a,b] \to D$ be a path. Assume each f_n is continuous. Then

$$\lim_{n \to \infty} \int_\gamma f_n(z)\, dz = \int_\gamma f(z)\, dz$$

(3) f_n is $\mathbb{C}$-diff. (holomorphic) for every n $\Rightarrow$ f is holomorphic and $\lim_{n\to\infty} f_n' = f$ uniformly.

Proof. (1) Let $a \in D$. We need to prove that $\lim_{z \to a} f(z) = f(a)$, that is, given $\varepsilon > 0$, we can find $\delta > 0$ such that $0 < |z-a| < \delta$ implies $|f(z) - f(a)| < \varepsilon$.

Write $f(z) - f(a) = f(z) - f_n(z) + f_n(a) - f(a) + f_n(z) - f_n(a)$

Pick $r > 0$ such that the closed disc centered at a, of radius r, is contained in D.

Let $n > 0$ be such that $|f_n(w) - f(w)| < \varepsilon/3$ for every w with $|w-a| \leq r$.

Let $\delta_1 > 0$ be such that $|f_n(z) - f_n(a)| < \varepsilon/3$ for every z with $|z-a| < \delta_1$ (exists since f_n is continuous).

Take $\delta = \min(r, \delta_1)$. Then we get (for $|z-a| < \delta$)

$$|f(z) - f(a)| \leq |f(z) - f_n(z)| + |f_n(a) - f(a)| + |f_n(z) - f_n(a)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \text{as desired.}$$

(2) We need to prove that for given $\varepsilon > 0$ we can find $N > 0$ so that

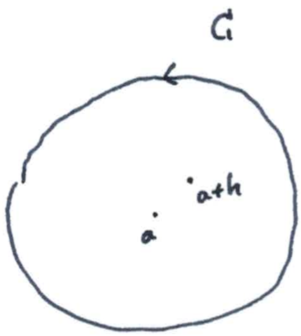
$$\left| \int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz \right| < \varepsilon \quad \text{for every } n \geq N.$$

Let L = length of γ. By definition of uniform continuity we can find N such that $|f_n(z) - f(z)| < \frac{\varepsilon}{L}$ for every $n \geq N$.

Then $$\left| \int_\gamma (f_n(z) - f(z))\,dz \right| \leq \frac{\varepsilon}{L} \cdot L = \varepsilon \quad \text{as required.}$$

(3) Let $a \in D$. We need to show that $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists

As before (see section (7.01)) pick $r > 0$ such that the closed disc of radius r centered at a, is in D. Let C = circle of radius r centered at a and assume, for definiteness, that $|h| < \frac{r}{2}$.



Then $$\frac{f(a+h) - f(a)}{h} = \lim_{n \to \infty} \frac{f_n(a+h) - f_n(a)}{h}$$

$$= \frac{1}{2\pi i} \lim_{n \to \infty} \frac{1}{h} \int_C \frac{f_n(z)}{z-a-h} - \frac{f_n(z)}{z-a}\,dz$$

(since each f_n is holomorphic)

$$= \frac{1}{2\pi i} \lim_{n\to\infty} \int_C \frac{f_n(z)}{(z-a-h)(z-a)}\, dz$$

Now $\left\{ f_n(z)/_{(z-a-h)(z-a)} \right\}$ converges uniformly on C to $\frac{f(z)}{(z-a-h)(z-a)}$. So, by

part (2)

$$\lim_{n\to\infty} \int_C \frac{f_n(z)}{(z-a-h)(z-a)}\, dz = \int_C \frac{f(z)}{(z-a-h)(z-a)}\, dz$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a-h)(z-a)}\, dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2}\, dz$$

(see the proof given in (7.0): it only assumes f is continuous).

and this completes the proof. □

In conclusion, uniform convergence implies that the order of operations $\lim_{n\to\infty}$, $\int$, $\frac{d}{dz}$ can be interchanged, which is not true in general.

Uniform Convergence of $\{f_n\}_{n=1,2,3,\ldots}$ $\Rightarrow$

$$\lim_{n\to\infty} \int_\gamma f_n(z)\, dz = \int_\gamma \lim_{n\to\infty} f_n(z)\, dz$$

$$\frac{d}{dz} \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{d}{dz} f_n(z)$$