Lecture 8

(8.0) Recall : last time we introduced the notion of uniform convergence of a sequence of functions $\{f_n\}$.

$\boxed{\begin{array}{l}\lim_{n\to\infty} f_n = f \\ \text{uniformly on a set } A\end{array}}$ means $\boxed{\begin{array}{l}\text{for every } \varepsilon>0, \text{ there is } N>0 \text{ such that} \\ |f_n(z)-f(z)|<\varepsilon \text{ for every } n\geq N \text{ and} \\ \text{for every } z\in A\end{array}}$

(8.1) Series. The notions of convergence / uniform convergence of a series $\sum_{n=1}^{\infty} c_n$ are the same for the sequence $\{S_n = c_1+c_2+\dots+c_n\}_{n=1,2,3,\dots}$

Just to spell it out :

A series of complex numbers $\sum_{n=1}^{\infty} c_n$ converges if for every $\varepsilon>0$ we can find $N>0$ such that $|c_{n+1}+\dots+c_{n+p}|<\varepsilon$ for every $n\geq N$ and every $p\geq 0$.

(This is just Cauchy's Criterion for convergence of corresponding sequence of partial sums $\{S_n = c_1+c_2+\dots+c_n\}_{n=1,2,3,\dots}$. Namely for every $\varepsilon>0$ there is $N>0$ such that $|S_{n+p}-S_n|<\varepsilon$ for every $n\geq N$ and $p\geq 0$.

And $S_{n+p} - S_n = c_{n+1}+c_{n+2}+\dots+c_{n+p}$)

Recall from Calculus II : ratio test and root test for convergence of a series.

Ratio test : if $\lim\limits_{n\to\infty} \dfrac{c_{n+1}}{c_n} < 1$ then the series $\sum\limits_{n=1}^{\infty} c_n$ converges.

Root test : if $\lim\limits_{n\to\infty} |c_n|^{1/n} < 1$ then the series $\sum\limits_{n=1}^{\infty} c_n$ converges.

An example of non-convergent (= divergent) series : $\sum\limits_{n=1}^{\infty} \frac{1}{n}$

Proof. Take ε to be less than $\frac{1}{2}$. Then no matter which N we pick

$$c_{N+1} + c_{N+2} + \dots + c_{2N} = \frac{1}{N+1} + \frac{1}{N+2} + \dots + \frac{1}{2N}$$

$$> \frac{1}{2N} + \frac{1}{2N} + \dots + \frac{1}{2N} = \frac{1}{2} > \varepsilon. \qquad \square$$

(8.2) Let $D \subset \mathbb{C}$ be a non-empty open set and $c_n : D \to \mathbb{C}$ functions. The series $\sum\limits_{n=1}^{\infty} c_n(z)$ is said to converge

- Pointwise : if for every $z_0 \in D$ and $\varepsilon > 0$, there exists $N > 0$ such that $|c_{n+1}(z_0) + \dots + c_{n+p}(z_0)| < \varepsilon$ for every $n \geq N$ and $p \geq 0$.

- Uniformly (on a subset $A \subset D$) : if for every $\varepsilon > 0$, there exists $N > 0$ such that $|c_{n+1}(z) + \dots + c_{n+p}(z)| < \varepsilon$ for every $n \geq N$, $p \geq 0$ and $z \in A$.

(8.3) An example of non uniform convergence. Consider the series

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots + \frac{x^2}{(1+x^2)^n} + \dots$$

$$
\begin{aligned}
S_n(x) &= \text{sum of first } n \text{ terms} \\
&= x^2\left[1 + (1+x^2)^{-1} + (1+x^2)^{-2} + \dots + (1+x^2)^{-n+1}\right] \\
&= x^2\left[\frac{1-(1+x^2)^{-n}}{1-(1+x^2)^{-1}}\right] = x^2\,\frac{(1+x^2)}{x^2}\,\frac{(1+x^2)^n-1}{(1+x^2)^n} \\
&= \frac{(1+x^2)^n-1}{(1+x^2)^{n-1}} = 1+x^2 - \frac{1}{(1+x^2)^{n-1}}
\end{aligned}
$$

For x real, $x \neq 0$ $\quad \lim_{n\to\infty} S_n(x) = 1+x^2$

For $x = 0$; $S_n(0) = 0$ and hence $\lim_{n\to\infty} S_n(0) = 0$

$\Big\}$ discontinuous at 0.

(8.4) Some examples of uniformly convergent series

Definition A series of the form $\sum_{n=0}^{\infty} c_n z^n$ is called a power series.
(here $c_n \in \mathbb{C}, n = 0, 1, 2, \dots$)

Abel's Theorem. Let $\sum_{n=0}^{\infty} c_n z^n$ be a power series. Then there exists $R \geq 0$ (R could possibly be ∞), such that $\sum_{n=0}^{\infty} c_n z^n$ is convergent for $|z| < R$ and divergent for $|z| > R$.

Moreover if $R \neq 0$, the convergence of $\sum_{n=0}^{\infty} c_n z^n$ on $\underbrace{\{w \in \mathbb{C} \text{ such that } |w| < R\}}^{D_R(0)}$ is uniform (on each compact subset of $D_R(0)$).

This number R is called the radius of convergence. Thus a power series $\sum_{n=0}^{\infty} c_n z^n$ defines a holomorphic function from $D_R(0)$ to $\mathbb{C}$.

(being a uniformly convergent limit of polynomial functions $\{c_0 + c_1 z + \dots + c_n z^n\}_{n=0,1,2,\dots}$)

Proof of Abel's Theorem.

Consider the set $S = \{ r \geq 0$ such that there is $M > 0$ with $|c_n| r^n \leq M$ for every $n \}$

S is a non-empty subset of the set of ~~positive~~ non-negative real numbers (because $0 \in S$)

and if $r \in S$ and $r' < r$ then $r' \in S$ (so S is an interval)

Let $R \geq 0$ be such that $S = [0, R]$ or $[0, R)$.

(1) if $|z| > R$ then $|c_n| |z|^n$ is not bounded. Meaning if we pick an $\varepsilon > 0$, there will be infinitely many n's with $|c_n| |z|^n > \varepsilon$.

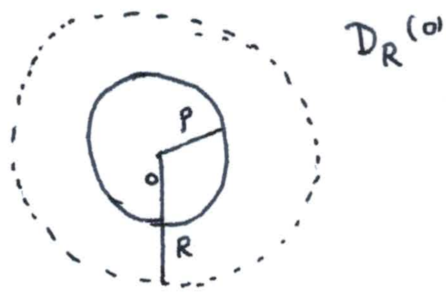
Hence $\sum_{n=0}^{\infty} c_n z^n$ is divergent.

(2) Now ~~let~~ assume that $R > 0$. We want to prove that $\sum_{n=0}^{\infty} c_n z^n$ converges uniformly on each compact set $K \subset D_R(0) = \{ w \in \mathbb{C}$ such that $|w| < R \}$

Pick $0 < \rho < R$ such that

$K \subset \{ w \in \mathbb{C}$ such that $|w| \leq \rho \}$

and let r be a number $\rho < r < R$.



By definition there is $M > 0$ such that

$|c_n| r^n \leq M$ for every n.

Hence for every $z \in K$, $|c_n z^n| \leq |c_n| \rho^n \leq \frac{M}{r^n} \rho^n = M \left(\frac{\rho}{r}\right)^n$

Since $\frac{\rho}{r} < 1$ we know that $\sum_{n=0}^{\infty} M \left(\frac{\rho}{r}\right)^n = \frac{M}{1 - \frac{\rho}{r}}$ converges.

$\Rightarrow \sum_{n=0}^{\infty} c_n z^n$ converges uniformly on K. $\square$

(8.5) Example 1. $1+z+z^2+z^3+\dots$ converges for $|z|<1$ $\left(1 = \text{radius of convergence}\right)$

diverges for $|z|>1$.

Proof. Let $|z| = r < 1$. Then $|z^{n+1} + \dots + z^{n+p}| \leq r^{n+1}(1+r+r^2+\dots+r^{p-1})$

(This is how ratio test works!)

$$= r^{n+1}\frac{(1-r^p)}{1-r} \leq \frac{r^{n+1}}{1-r}$$

Given $\varepsilon > 0$, pick $N > 0$ such that $r^{N+1} < \varepsilon(1-r)$.

Then for every $n \geq N$ and $p \geq 1$ we get

$|z^{n+1} + \dots + z^{n+p}| \leq \frac{r^{n+1}}{1-r} \leq \frac{r^{N+1}}{1-r} < \varepsilon$. $\Rightarrow \sum_{n=0}^{\infty} z^n$ converges for $|z|<1$

If $|z| = r > 1$, then take (for instance) $\varepsilon = 1$. Now for every $n \geq 0$

$|z|^n = r^n > \varepsilon$ and hence $\sum_{n=0}^{\infty} z^n$ diverges. $\square$

(8.6) Example 2.

$$e^z := 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$ converges for every z.

(radius of convergence $= \infty$)

Proof

Let $R = |z|$.

Ratio test: $\left|\frac{c_{n+1}}{c_n}\right| = \frac{|z|^{n+1}}{(n+1)!}\frac{n!}{|z|^n} = \frac{R}{n+1} \to 0$ as $n \to \infty$. (<1)

Hence $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is convergent. $\square$

(8.7) Example 3. Let $f: D \to \mathbb{C}$ be a holomorphic function.

Let $a \in D$ and let $r > 0$ be such that

$$\{ w \in \mathbb{C} \text{ such that } |w-a| \leq r \} \subset D.$$

Then $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$ has non-zero

radius of convergence ($\geq r$).

Proof. By Cauchy's formula $\displaystyle\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$. Then

Let $M > 0$ be such that $|f(z)| \leq M$ for every z on C.

$$\left| \frac{f^{(n)}(a)}{n!} \right| \leq \frac{1}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{M}{r^n}.$$

Hence, for z such that $\underset{\rho}{\underset{\parallel}{|z-a|}} < r$, we get

$$\left| \frac{f^{(n)}(a)}{n!} (z-a)^n \right| \leq M \left(\frac{\rho}{r}\right)^n \quad \text{and} \quad \sum_{n=0}^{\infty} M\left(\frac{\rho}{r}\right)^n = \frac{M}{1-\rho/r}$$

converges

$\square$

(8.8) Algebraic operations on power series.

Let $\displaystyle c(z) = \sum_{n=0}^{\infty} c_n z^n$ and $\displaystyle d(z) = \sum_{n=0}^{\infty} d_n z^n$ be power series with radii of convergence R_1 and R_2 respectively. Then

(1) $\displaystyle c(z) + d(z) := \sum_{n=0}^{\infty} (c_n + d_n) z^n$ has radius of convergence $\geq \min(R_1, R_2)$

(2) $$c(z)\,d(z) := \sum_{n=0}^{\infty} \left(c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \dots + c_n d_0\right) z^n$$

(radius of convergence $\geq \min(R_1, R_2)$)

Example. $$e^a e^b = e^{a+b}$$

Proof. Recall the binomial expansion:

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} a^k b^{n-k}.$$ Using this we get

$$e^a e^b = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a^k}{k!} \frac{b^{n-k}}{(n-k)!} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \frac{n!\, a^k b^{n-k}}{k!\,(n-k)!} \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (a+b)^n = e^{a+b}$$

(8.9) Using the expansion of $\sin(x)$ and $\cos(x)$ for $x \in \mathbb{R}$ we get

$(y \in \mathbb{R})$

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots$$

$$= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \frac{y^8}{8!} + \cdots$$

$$+ i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots \right)$$

$$= \cos(y) + i \sin(y)$$

Hence $$\boxed{e^{x+iy} = e^x \left(\cos(y) + i \sin(y) \right)}$$

e.g. $e^{\pi i} = -1$ $\quad e^{2\pi k i} = 1$ for every $k \in \mathbb{Z}$.

Definition of $\sin(z)$ and $\cos(z)$ – just by power series

$$\left.\begin{aligned} \cos(z) &:= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \\ \sin(z) &:= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \end{aligned}\right\} \text{have } \infty \text{ radius of convergence}$$

Note: $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$ $\qquad \sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$

(8.10) Important consequence of Uniform Convergence and Abel's Theorem:

$$\boxed{\text{Power series can be differentiated and integrated termwise}}$$

Reason is simple. Clearly polynomials can be differentiated and integrated termwise. By Abel's Theorem, within its disc of convergence, a power series is a uniform limit of polynomials. By Theorem (7.3) it follows that power series can also be termwise differentiated and integrated:

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} c_n z^n\right) = \frac{d}{dz} \overset{\text{uniform by Abel's Theorem}}{\lim_{n\to\infty}} (c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n)$$

$$= \lim_{n\to\infty} \frac{d}{dz}(c_0 + c_1 z + \ldots + c_n z^n) = c_1 + 2c_2 z + 3c_3 z^2 + \ldots$$

(Theorem (7.3))

$$= \sum_{n=1}^{\infty} n c_n z^{n-1}$$

Summary: $\sum_{n=0}^{\infty} c_n z^n$ defines a holomorphic function on domain $D = \{ w \in \mathbb{C}$ such that $|w| < R \}$

(R = radius of convergence)

Conversely a holomorphic function near $a \in \mathbb{C}$ admits power series representation (Taylor Series of f)

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

(the radius of convergence of the right-hand side will be the largest number R such that the disc of radius R centered at a is contained in the domain of f).