Lecture 9

(9.0) Recall: we introduced the notion of power series and radius of convergence: given a power series $\sum_{n=0}^{\infty} c_n z^n$ there is a unique positive (or zero, or infinity) real number R such that

- $\sum_{n=0}^{\infty} c_n z^n$ converges uniformly (on compact sets contained in) $D_R(0)$

$$D_R(0) = \{ w \in \mathbb{C} \text{ such that } |w| < R \}$$

- for $|z| > R$ $\sum_{n=0}^{\infty} c_n z^n$ diverges

Thus $f(z) = \sum_{n=0}^{\infty} c_n z^n : D_R(0) \to \mathbb{C}$ is a holomorphic function. Moreover, using Theorem (7.3) of Lecture 7, power series can be differentiated termwise

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1} \Rightarrow f'(0) = c_1$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) c_n z^{n-2} \Rightarrow f''(0) = 2c_2$$

$$\vdots$$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) c_n z^{n-k} \Rightarrow f^{(k)}(0) = k!\, c_k$$

Therefore $c_k = \dfrac{f^{(k)}(0)}{k!}$. So $\boxed{f(z) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} z^n}$ (Taylor series)

Slightly more generally we can have power series centered at $a \in \mathbb{C}$:

$\sum_{n=0}^{\infty} c_n (z-a)^n$ will define a holomorphic function $f: D_R(a) \to \mathbb{C}$

with $c_n = \dfrac{f^{(n)}(a)}{n!}$.

(9.1) As a few examples we defined

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (\text{radius of convergence} = \infty)$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

radius of convergence $= \infty$

Thus e^z, $\sin(z)$, $\cos(z)$ are holomorphic functions on domain $= \mathbb{C}$.

A holomorphic function defined on $\mathbb{C}$ is called <u>entire</u> function.

Ex. $\frac{d}{dz} e^z = e^z \qquad \frac{d}{dz} \sin(z) = \cos(z) \qquad \frac{d}{dz}\cos(z) = -\sin(z)$

We also proved last time that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \qquad (\theta \in \mathbb{R})$$

$$e^{x+iy} = e^x(\cos(y) + i\sin(y)) \qquad (x, y \in \mathbb{R})$$

Ex. Recall multiplication of power series

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{l=0}^{\infty} d_l z^l\right) = \sum_{n=0}^{\infty} (c_0 d_n + c_1 d_{n-1} + \cdots + c_n d_0) z^n$$

Use this to check that

$$\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

Alternately $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$ and $e^{z+w} = e^z \cdot e^w$ can be used to prove this.

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

(9.2) Logarithm.

Since $e^{x+iy} = e^{x}(\cos(y) + i\sin(y))$ for every $x, y \in \mathbb{R}$.,

its inverse must be given by

$$\text{"log"}(w) = \ln|w| + i\arg(w)$$

. Unfortunately, given non-zero $w \in \mathbb{C}$, $\arg(w)$ is only defined up to adding an integer multiple of 2π.

We already know that there can never be a holomorphic function

$$\text{"log"} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$$

which is inverse of e^{z}.

Proof. Assume there is. That is $e^{\text{"log"}(z)} = z$ for every $z \in \mathbb{C}$, $z \neq 0$.

Then $\frac{d}{dz}\left(e^{\text{"log"}z}\right) = \frac{d}{dz}(z) = 1$

By chain rule (see Lecture 2 for a proof)

$$\frac{d}{dz}\left(e^{\text{"log"}z}\right) = e^{\text{"log"}(z)}\frac{d}{dz}(\text{"log"}z) = z\frac{d}{dz}\text{"log"}(z)$$

$\Rightarrow \frac{d}{dz}\text{"log"}z = \frac{1}{z}$. Meaning $\frac{1}{z}$ would have an antiderivative on the domain $\mathbb{C} \setminus \{0\}$. But that would mean $\int_\gamma \frac{1}{z}dz = 0$ for every closed path γ, by Theorem (3.5) of Lecture 3.

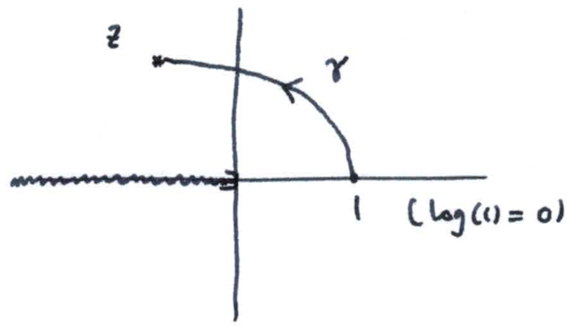
That is not true $\int \frac{1}{z}dz = 2\pi i \neq 0$ $\square$

☹

Define log on the domain $D = \{ z \in \mathbb{C}$ such that z is not a negative real number, or zero$\}$

by :

$$\boxed{\log(z) = \ln|z| + i\arg(z) \quad \text{where} \quad -\pi < \arg(z) < \pi}$$



Picture of D : remove negative real line and 0.

~~for~~ Alternately

$$\log z = \int_{\gamma} \frac{1}{w}\, dw$$

γ = a path joining 1 and z

By Morera's Theorem $\log : D \to \mathbb{C}$ is a holomorphic function; since for every simple closed path in D, say $\gamma : [a,b] \to D$,

$$\int_{\gamma} \frac{1}{z}\, dz = 0$$

(since γ cannot enclose 0, $\frac{1}{z}$ is holomorphic on $\mathrm{Int}(\gamma)$ and Cauchy's theorem applies)

(9.3) Taylor Series expansion of $\log(z)$ near $z = 1$:

$$\frac{d}{dz}\log(z) = \frac{1}{z} \qquad \left(\frac{d}{dz}\right)^2 \log(z) = \frac{-1}{z^2}$$

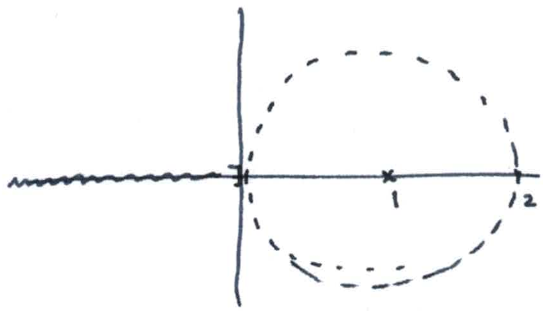
$$\left(\frac{d}{dz}\right)^3 \log(z) = \frac{2}{z^3} \qquad \dots \qquad \left(\frac{d}{dz}\right)^n \log(z) = (-1)^{n-1}\,\frac{2\cdot 3\cdot 4 \cdots (n-1)}{z^n}$$

$$\Rightarrow \quad \frac{1}{n!}\left[\left(\frac{d}{dz}\right)^n \log(z)\right]_{\text{set } z=1} = (-1)^{n-1}\frac{(n-1)!}{n!} = (-1)^{n-1}\frac{1}{n}. \quad (n \geq 1)$$

$$\log(z) = 1.(z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 \ldots\ldots$$

$$= \sum_{n\geq 1}^{\infty} (-1)^{n-1}\frac{(z-1)^n}{n}$$

Radius of convergence = 1



Write $z = 1+w$ to get power series centered at 0

$$\boxed{\log(1+w) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{w^n}{n} \quad \text{for } |w|<1}$$

(9.4) Another example: write Taylor series of $\frac{1}{z^2-5z+6}$ near $z=1$.

$$\frac{1}{z^2-5z+6} = \frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2}$$

$$\frac{1}{z-3} = \frac{1}{(z-1)-2} = -\frac{1}{2-(z-1)} = \frac{-1}{2}\left(\frac{1}{1-\frac{z-1}{2}}\right)$$

$$= \frac{-1}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} \quad \text{for } \left|\frac{z-1}{2}\right| < 1$$

Similarly
$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = -\frac{1}{1-(z-1)}$$
$$= -\sum_{n=0}^{\infty} (z-1)^n \qquad \text{for } |z-1|<1$$

Hence
$$\frac{1}{z^2-5z+6} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} - \left(-\sum_{n=0}^{\infty}(z-1)^n\right) \qquad \text{for } |z-1|<1$$
$(\min(1,2))$
$$= \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)(z-1)^n$$

(9.5) Application of Power Series : Identity Theorem

Let $D \subset \mathbb{C}$ be an open, connected set and $f : D \to \mathbb{C}$ a holomorphic function. Given $b \in D$, there is $r>0$ such that
$$f(z) = \sum_{n=0}^{\infty} c_n (z-b)^n \quad \text{on } D_b(r) = \text{open disc of radius } r \text{ centered at } b.$$

There are two possibilities

(A) $c_n = 0$ for every n, hence $f(z)=0$ for every z such that $|z-b|<r$

(B) $c_0 = c_1 = \dots = c_{N-1} = 0$ and $c_N \neq 0$ (for some $N \geq 0$)
$$f(z) = (z-b)^N \boxed{\sum_{n=0}^{\infty} c_{N+n} (z-b)^n} = g(z)$$
$$f(z) = (z-b)^N g(z) \quad \text{where } g(b) = c_N \neq 0.$$

In possibility (B) we can find perhaps a smaller radius, say r_1, $0 < r_1 < r$, such that $g(z) \neq 0$ for any z with $|z-b| < r_1$.

Proof. Otherwise, for every $n \geq 2$, there will be a point $b_n \in D_{\frac{r}{n}}(b)$ where $g(b_n) = 0$.

But then $\lim_{n\to\infty} b_n = b$ (by definition) and hence by continuity $g(b) = \lim_{n\to\infty} g(b_n) = 0$ contradicting $g(b) = c_N \neq 0$ □

If (B) is the case, we say <u>b is an (isolated) zero of $f(z)$ of order N (assuming $N \geq 1$).</u>

Theorem. Assume there exists a sequence $\{a_n\}_{n=1,2,3,\dots}$ in D such that $\lim_{n\to\infty} a_n$ exists and $a = \lim_{n\to\infty} a_n \in D$.

If $f(a_n) = 0$ for every $n = 1, 2, 3, \dots$ (and hence, by continuity of f, $f(a) = 0$) then $f(z) = 0$ for every $z \in D$.

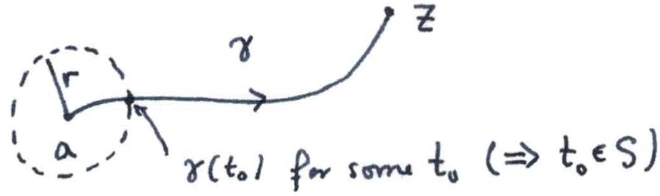
As a consequence of this, if f and $\tilde{f}$ are two holomorphic functions on D, such that $f(a_n) = \tilde{f}(a_n)$ for some convergent sequence in D $\left(\{a_n\},\ a = \lim_{n\to\infty} a_n;\ a, a_2, a_2, \dots \in D\right)$ then $f = \tilde{f}$ on D. This is because $f - \tilde{f}$ is zero on $a, a_1, a_2, \dots$ and by Theorem above

$$(f - \tilde{f})(z) = f(z) - \tilde{f}(z) = 0 \quad \text{for every } z \in D.$$

Proof of Theorem: Let $z \in D$ and let $\gamma : [0,1] \to D$ be a simple path with $\gamma(0) = a$ and $\gamma(1) = z$ (exists since D is connected).

Let $S \subset [0,1]$ defined as

$$S = \{ t \text{ such that for every } 0 \le t' < t \quad f(\gamma(t')) = 0 \}$$



(i.e. f is zero on $\gamma([0,t))$). Since $f(a_n) = 0$, $a_n \to a$ as $n \to \infty$,

For $a \in D$, only possibility is A, that is, there exists $r > 0$ such that $f(z) = 0$ for every $|z - a| < r$. This implies that the set S is non-empty. Moreover if $t \in S$ and $t' < t$ then $t' \in S$ by definition.

Hence $S \subset [0,1]$ is an interval.

It remains to show that $S = [0,1]$. Otherwise, $S = [0,T]$ and for $b = \gamma(T)$, $f(b) = 0$ and we are back in possibility A above, implying that there is $r > 0$ such that $T + r \in S$.

Hence $1 \in S$

$$\Rightarrow \quad f(\gamma(1)) = f(z) = 0$$

as claimed

$\square$