Lecture 0: Introduction

(0.0) The aim of this course is to study the structure and representation theory of three classes of affine (or infinite-dimensional) quantum groups.

Yangians	Quantum loop algebras	Elliptic quantum groups.
$Y_h\mathfrak{g}$	$\mathcal{U}_q(L\mathfrak{g})$	$\mathcal{E}_{h,\tau}(\mathfrak{g})$

These are Hopf algebras associated to a simple Lie algebra $\mathfrak{g}$. Most importantly they come equipped with an R-matrix which satisfies Yang-Baxter equation. We will also study (time permitting) several difference-differential equations related to these.

Today's lecture is aimed at explaining the origins of these objects for $\mathfrak{g} = \mathfrak{sl}_2$. The plan is as follows:

Lattice models of Statistical Mechanics

$\downarrow$

Yang-Baxter equation = Solvability Criteria

Three solutions of YBE

$\swarrow$ $\downarrow$ $\searrow$

Rational | Trigonometric | Elliptic

RTT algebras of Faddeev-Reshetikhin-Takhtajan

$\downarrow$ $\downarrow$ $\downarrow$

$Y_h\mathfrak{sl}_2$ | $\mathcal{U}_q(L\mathfrak{sl}_2)$ | $\mathcal{E}_{h,\tau}(\mathfrak{sl}_2)$

Finally I will explain how to obtain a new presentation (Drinfeld's) of these algebras ($Y_\hbar sl_2$ & $U_q(Lsl_2)$); through Gaussian dec. This is the presentation that can be generalized to arbitrary $\mathfrak{g}$ and will be the focus of our course.

(0.1) Lattice Models. Let $M, N \in \mathbb{N}^\times$ and let $\Lambda_{M,N}$ be a grid (rectangular) with M rows and N colums. We impose periodic boundary condition, so $\Lambda_{M,N}$ is infact a grid on a torus.

A <u>configuration C</u> is an assignment of $+$ or $-$ on the edges of $\Lambda_{M,N}$.

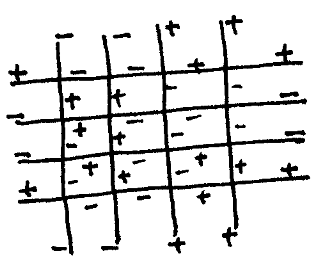
Thus around each vertex there are 16 possible assignments.

Let $a_1, \ldots, a_{16}$ be arbitrary (complex) numbers. <u>Weight of a configuration</u> C is

$$w(C) = \prod_{i=1}^{16} a_i^{m_i} \qquad m_i = \# \text{ of vertices in } i^{th} \text{ possibility.}$$

Partition function $\displaystyle Z = \sum_{C:\text{config}} w(C)$.

e.g.



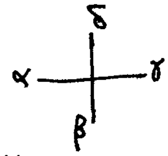
example of a configuration on $\Lambda_{4,4}$.

In Statistical Mechanics, people are intersted in computing Z (and expectation values etc.) or atleast there behavior as $M, N \to \infty$.

We realize Z as trace of a <u>product</u> (power) of a matrix called Transfer matrix.

(0.2) R-matrix and T-matrix.

We encode $a_1 \dots a_{16}$ in a matrix $R = \left(R(\alpha\beta|\gamma\delta)\right)_{\alpha,\beta,\gamma,\delta \in \{\pm\}}$

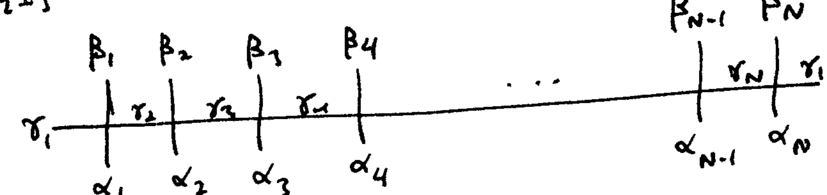
$R(\alpha\beta|\gamma\delta)$ is the weight of 

Notation $\alpha, \beta, \gamma, \delta, \dots$ greek letters $\in \{\pm\}$

$\underline{\alpha}, \underline{\beta}, \dots \quad \in \{\pm\}^N \qquad \underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_N)$.

Transfer matrix T. For $\underline{\alpha}, \underline{\beta} \in \{\pm\}^N$ define

$$T(\underline{\alpha}, \underline{\beta}) = \sum_{\underline{\gamma} \in \{\pm\}^N} R(\gamma_1\alpha_1|\gamma_2\beta_1)\, R(\gamma_2\alpha_2|\gamma_3\beta_2) \dots R(\gamma_N\alpha_N|\gamma_1\beta_N)$$



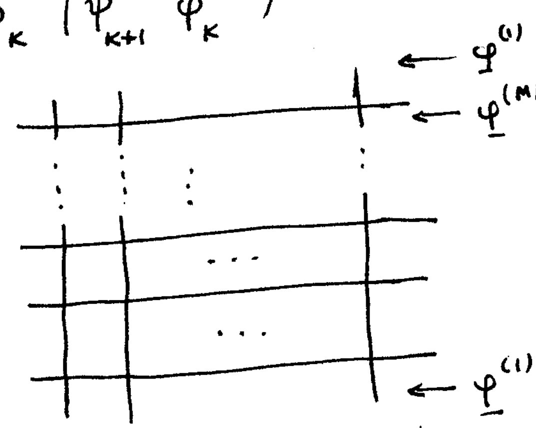
We can think of R as an operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and T as an operator on $(\mathbb{C}^2)^{\otimes N}$.

Lemma. $Z = \mathrm{Tr}(T^M)$

Proof. $\mathrm{Tr}(T^M) = \displaystyle\sum_{\underline{\varphi}^{(1)}, \dots, \underline{\varphi}^{(M)}} T(\underline{\varphi}^{(1)}|\underline{\varphi}^{(2)}) \dots T(\underline{\varphi}^{(M)}|\underline{\varphi}^{(1)})$

$$= \sum_{\underline{\varphi}^{(1)} \dots \underline{\varphi}^{(M)}} \; \sum_{\underline{\psi}^{(1)} \dots \underline{\psi}^{(M)}} \; \prod_{\substack{i=1\dots M \\ k=1\dots N}} R(\psi_k^{(i)} \varphi_k^{(i)} | \psi_{k+1}^{(i)} \varphi_k^{(i+1)})$$

$$= \sum_{C} w(C) = Z \qquad \square$$

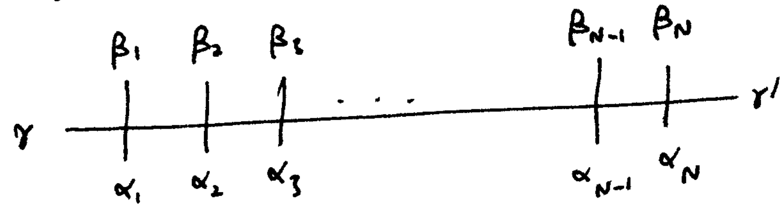


In fact, T can be thought of as trace of another matrix, called the monodromy matrix $\mathcal{T}$.

$\mathcal{T} : \mathbb{C}^2_0 \otimes (\mathbb{C}^2)^{\otimes N} \circlearrowleft$ is defined analogously.

$$\mathcal{T}(\gamma\, \underline{\alpha} \,|\, \gamma' \underline{\beta}) = \sum_{\gamma_2, \dots, \gamma_N \in \{\pm\}} R(\gamma\alpha_1 | \gamma_2 \beta_1) \quad \dots \quad R(\gamma_N \alpha_N | \gamma' \beta_N)$$

$\mathcal{T} = R_{01} R_{02} \dots R_{0N}$



Then $T = \mathrm{Tr}_{\mathbb{C}^2_0}(\mathcal{T})$.

(0.3) Commuting Transfer matrices.

T = transfer matrix corr. to $(a_1, \dots, a_{16})$ similarly R, R', R''.

T' = " " " $(a'_1, \dots, a'_{16})$

Let us obtain (necessary and) sufficient conditions for $TT' = T'T$

$$TT' = \mathrm{Tr}_{\mathbb{C}^2_0 \otimes \mathbb{C}^2_{\bar{0}}}(\mathcal{T}_0 \mathcal{T}'_{\bar{0}})$$

$$T'T = \mathrm{Tr}_{\mathbb{C}^2_0 \otimes \mathbb{C}^2_{\bar{0}}}(\mathcal{T}'_{\bar{0}} \mathcal{T}_0)$$

If $R''_{0\bar{0}}$ conjugates $\mathcal{T}_0 \mathcal{T}'_{\bar{0}}$ to $\mathcal{T}'_{\bar{0}} \mathcal{T}_0$, we will have $TT' = T'T$.

$$R''_{0\bar{0}} \mathcal{T}_0 \mathcal{T}'_{\bar{0}} = \mathcal{T}'_{\bar{0}} \mathcal{T}_0 R''_{0\bar{0}} \in \mathrm{End}\left(\mathbb{C}^2_0 \otimes \mathbb{C}^2_{\bar{0}} \otimes (\mathbb{C}^2)^{\otimes N}\right)$$

For $N = 1$ we get

$$R''_{0\bar{0}} R_{01} R'_{\bar{0}1} = R'_{\bar{0}1} R_{01} R''_{0\bar{0}}$$ Yang-Baxter equation (YBE)

<u>Thm</u>. (YBE) $\Rightarrow$ $TT' = T'T$

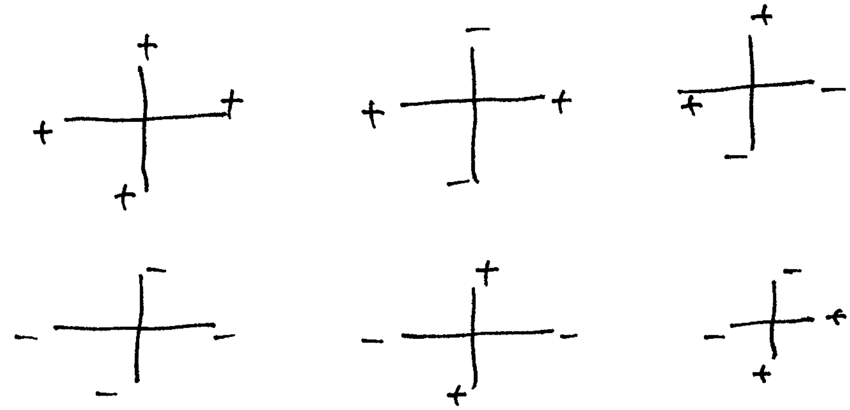
Proof. $R''_{0\bar{0}}\, T_0\, T'_{\bar{0}} = R''_{0\bar{0}}\, R_{01} \cdots R_{0N}\, R'_{\bar{0}1} \cdots R'_{\bar{0}N}$

$= R'_{\bar{0}1}\, R_{01}\, R''_{0\bar{0}}\, R_{02} \cdots R_{0N}\, R'_{\bar{0}2} \cdots R'_{\bar{0}N}$

$\cdots = T'_{\bar{0}}\, T_0\, R''_{0\bar{0}}$ $\square$.

(0.4) 6 vertex model.

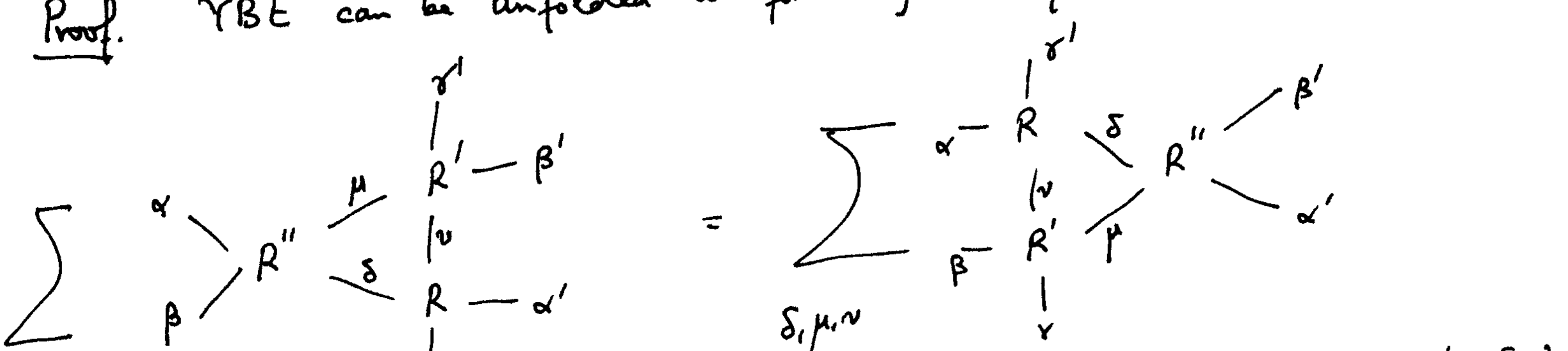
Allowed configurations



a b c

For 6v model YBE $\Rightarrow$ $\dfrac{a^2+b^2-c^2}{2ab} = \dfrac{a'^2+b'^2-c'^2}{2a'b'}$ $(= \Delta$ say$)$

Proof. YBE can be unfolded to following 64 equations



$\forall\ \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \{\pm\}$

$$\sum_{\delta,\mu,\nu} R''(\alpha\beta|\delta\mu)\, R(\delta\gamma|\alpha'\nu)\, R'(\mu\nu|\beta'\gamma') = \sum_{\delta,\mu,\nu} R'(\beta\gamma|\mu\nu)\, R(\alpha\nu|\delta\gamma')\, R''(\delta\mu|\alpha'\beta')$$

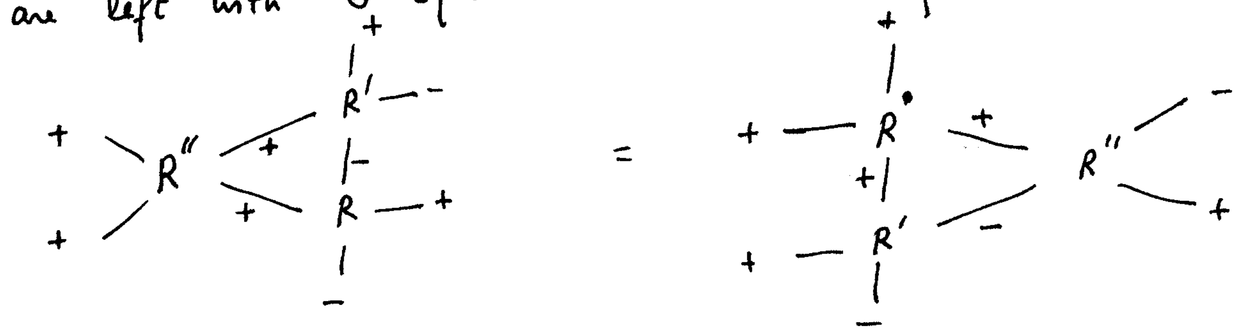
6v case $\Rightarrow$ both sides are zero unless $\alpha+\beta+\gamma = \alpha'+\beta'+\gamma'$

$\to$ 20 equations $\xrightarrow{\text{sign symmetry}}$ 10 equations

If $\alpha = \alpha'$ $\beta = \beta'$ $\gamma = \gamma'$ the equation holds trivially since R is symmetric.

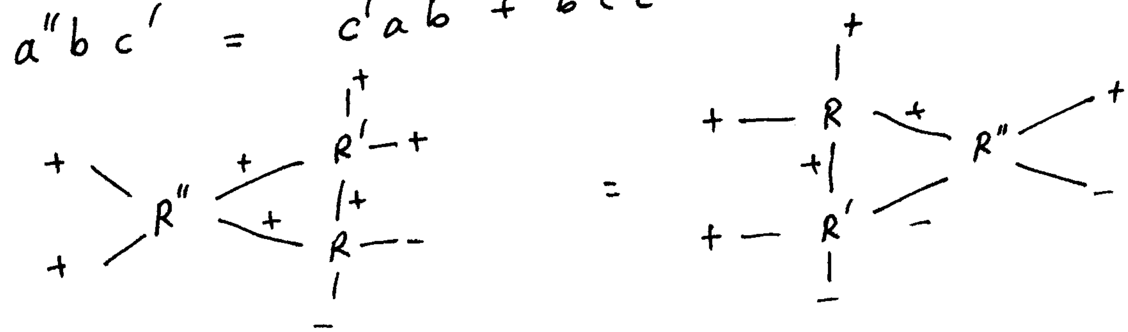
We are left with 6 equations which come in pairs.

1



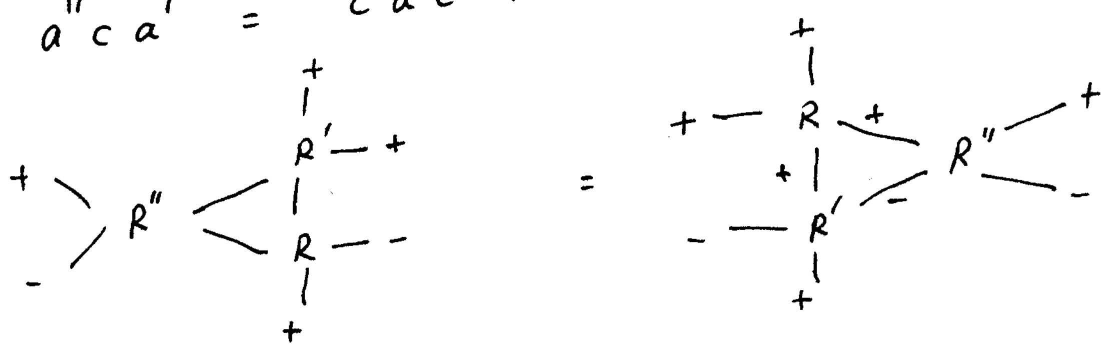
$$a''bc' = c'ab'' + b'cc''$$

2



$$a''ca' = c'ac'' + b'cb''$$

3



$$c''ba' = b'ac'' + c'cb''$$

Eliminate a'' b'' and c'' to get $\dfrac{a^2+b^2-c^2}{2ab} = \dfrac{a'^2+b'^2-c'^2}{2a'b'}$

Use 1 & 2 to eliminate a'' :

$$ca'(c'ab'' + b'cc'') = bc'(c'ac'' + b'cb'')$$

. Use 3 $b''cc' = a'bc'' - ab'c''$ cancel c''.

$$\Rightarrow b''(aa'cc' - bb'cc') = c''(ab(c')^2 - a'b'c^2)$$

$$(aa' - bb')(a'b - ab') = ab(c')^2 - a'b'c^2$$

$$\Rightarrow aa'^2b - a^2a'b' - a'b^2b' + abb'^2 = abc'^2 - a'b'c^2$$

$$\Rightarrow (a^2+b^2-c^2)a'b' = (a'^2+b'^2-c'^2)ab \qquad \square$$

(0.5) Rational solution. Let $\Delta = 1$ so that

$$(a-b)^2 = c^2 \qquad a = b+c$$

Let $c = \hbar$ (fixed) $b = u$ and hence $a = u + \hbar$. We get

$$R(u) = \begin{bmatrix} u+\hbar & 0 & 0 & 0 \\ 0 & u & \hbar & 0 \\ 0 & \hbar & u & 0 \\ 0 & 0 & 0 & u+\hbar \end{bmatrix}$$

satisfying $R_{12}(u)\,R_{13}(u+v)\,R_{23}(v)$
$= R_{23}(v)\,R_{13}(u+v)\,R_{12}(u)$

Trigonometric soln $\quad \dfrac{a^2+b^2-c^2}{2ab} = \Delta$

$$\Rightarrow \quad 1 + (b/a)^2 - (c/a)^2 = 2\,(b/a)\,\Delta$$

Let $x = b/a$ and $y = c/a$

$$y^2 = x^2 - 2x\Delta + 1 = (x-q)(x-q^{-1}) \qquad \text{where } \Delta = \frac{q+q^{-1}}{2}$$

Let $t = \left(\dfrac{x-q^{-1}}{x-q}\right)^{\frac{1}{2}} \Rightarrow x = \dfrac{qt - q^{-1}t^{-1}}{t - t^{-1}}$

$y = (x-q)\,t \Rightarrow y = \dfrac{q - q^{-1}}{t - t^{-1}}$

$a = t - t^{-1}$
$b = qt - q^{-1}t^{-1}$
$c = q - q^{-1}$

$a = q^{-1}\zeta - q\zeta^{-1}$ where $t = q^{-1}\zeta$
$b = \zeta - \zeta^{-1}$
$c = q - q^{-1}$

$$R_{12}(\zeta)\,R_{13}(\zeta\zeta')\,R_{23}(\zeta') = R_{23}(\zeta')\,R_{13}(\zeta\zeta')\,R_{12}(\zeta)$$

(0.6) RTT algebra (rational case)

Let Y be an algebra generated by $\{t_{ij}^{(r)}\}_{\substack{1\le i,j\le 2 \\ r\in\mathbb{N}}}$.

$t_{ij}(u) = \delta_{ij} + \hbar \sum_{r\ge 0} t_{ij}^{(r)} u^{-r-1}$. We impose the relations

$$R(u-v)\, T^{(1)}(u)\, T^{(2)}(v) = T^{(2)}(v)\, T^{(1)}(u)\, R(u-v)$$

These relations can be unfolded to

$$[t_{ij}(u), t_{k\ell}(v)] = \frac{\hbar}{u-v}\left(t_{kj}(u)\, t_{i\ell}(v) - t_{kj}(v)\, t_{i\ell}(u)\right)$$

Natural Hopf structure $\Delta(t_{ij}) = \sum_{k=1}^{2} t_{ik} \otimes t_{kj}$.

This algebra turns out to be the Yangian of $\mathfrak{gl}_2$. To get $\mathfrak{sl}_2$ we must set $q\mathrm{Det}(T(u)) := t_{11}(u)\, t_{22}(u-\hbar) - t_{21}(u)\, t_{12}(u-\hbar) = 1$.

Drinfeld's new presentation can be obtained as follows.

$$T(u) = \begin{bmatrix} 1 & 0 \\ x^-(u) & 1 \end{bmatrix} \begin{bmatrix} \xi_1(u) & 0 \\ 0 & \xi_2(u) \end{bmatrix} \begin{bmatrix} 1 & x^+(u) \\ 0 & 1 \end{bmatrix}$$

Thus Y can be presented on generators $\{\xi_{1,r}, \xi_{2,r}, x_r^{\pm}\}_{r\in\mathbb{N}}$ which is closer to the presentation of Kac-Moody algebras and can be generalized.

(0.7) RTT algebra (trigonometric case)

Let $\mathcal{U}$ be an algebra generated by $\{\ell_{ij}^{(r)}, \bar{\ell}_{ij}^{(r)}\}_{r\in\mathbb{N}}$

$$\ell_{ij}^{+}(z) = \sum_{r\ge 0} \ell_{ij}^{(r)} z^{-r} \qquad \ell_{ij}^{-}(z) = \sum_{r\ge 0} \bar{\ell}_{ij}^{(r)} z^{r}$$

Relations : $\quad l_{ij}^{(0)} = \bar{l}_{ji}^{(0)} \; (=0) \quad \forall \; i<j$

$$l_{ii}^{(0)} = \left(\bar{l}_{ii}^{(0)}\right)^{-1}.$$

$$R(z w^{-1}) \, L^{\varepsilon_1,(1)}(z) \, L^{\varepsilon_2,(2)}(w) = L^{\varepsilon_2,(2)}(w) \, L^{\varepsilon_1,(1)}(z) \, R(z w^{-1})$$

for $\varepsilon_1, \varepsilon_2 \in \{\pm\}$.

This turns out to be quantum loop algebra of gl_2. One can similarly obtain its presentation on $\{H_{1,k}, H_{2,k}, X_k^{\pm}\}_{k \in \mathbb{Z}}$ (Drinfeld's new presentation).