Lecture 1. Kac-Moody algebras

(1.0) A Lie algebra $\mathfrak{g}$ is a vector space (over $\mathbb{C}$) together with a bilinear pairing $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[x,y] = -[y,x] \qquad\qquad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

skew-symmetry $\qquad\qquad$ Jacobi identity

A representation of $\mathfrak{g}$ is a linear map $\mathfrak{g} \xrightarrow{\rho} \mathrm{End}(V)$ s.t.

$$\rho[x,y] = \rho(x)\rho(y) - \rho(y)\rho(x)$$

Note: Jacobi identity is equivalent to saying that $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a representation, where $\mathrm{ad}(x)\cdot y = [x,y]$.

Universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is defined as the following unital associative algebra

$$T(\mathfrak{g}) = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$$

$$\mathcal{U}(\mathfrak{g}) = T\mathfrak{g} \big/ \langle x \otimes y - y \otimes x = [x,y] \rangle \qquad \text{(tensor algebra)}$$

(1.1) We are interested in a class of Lie algebras called Kac-Moody algebras. These are algebras associated to an integer matrix,

Cartan Matrix $\quad A = (a_{ij})_{i,j \in I} \quad$ I : finite indexing set

- $a_{ii} = 2$
- $a_{ij} \in \mathbb{Z}_{\leq 0} \quad \forall i \neq j$
- there exists a diagonal matrix $D = (d_i)_{i \in I}$ with $d_i \in \mathbb{N}^\times$ s.t. $d_i a_{ij} = d_j a_{ji} \quad \forall i,j$

(additionally assume $\gcd(d_i : i \in I) = 1$).

Further let A be indecomposable, i.e. $\nexists$ non-trivial $I = I_1 \sqcup I_2$ s.t. $a_{ij} = 0 \quad \forall\, i \in I_1, j \in I_2$.

A <u>realization of A</u> is a vector space $\mathfrak{h}$ of dim $2|I| - \mathrm{rank}(A)$ together with linearly independent sets $\Delta = \{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$ $\check{\Delta} = \{h_i\}_{i\in I} \subset \mathfrak{h}$ such that $\alpha_i(h_j) = a_{ji}$ $\forall\, i,j \in I$.

α_i's are called simple roots. h_i's are called simple coroots.

$Q = \sum_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ (root lattice) $\quad Q^\vee = \sum_{i\in I} \mathbb{Z} h_i \subset \mathfrak{h}$ coroot lattice

Define $\tilde{\mathfrak{g}}(A)$ to be Lie algebra generated by $h\in\mathfrak{h}$, e_i, f_i $(i\in I)$ subject to

$[h,h'] = 0 \quad [h,e_i] = \alpha_i(h)e_i \quad [h,f_i] = -\alpha_i(h) f_i \quad [e_i, f_j] = \delta_{ij} h_i$

Basic facts: (i) $\tilde{\mathfrak{g}}(A) = \tilde{n}_- \oplus \mathfrak{h} \oplus \tilde{n}_+$ as vector spaces.

where $\tilde{n}_\pm$ are free Lie algebras generated by e_i and f_i $(i\in I)$ resp.

(ii) $\tilde{n}_\pm = \bigoplus_{\substack{\alpha\in Q_+\\ (\alpha\neq 0)}} \tilde{\mathfrak{g}}_{\pm\alpha}$ where for $\gamma\in\mathfrak{h}^*$ $\tilde{\mathfrak{g}}_\gamma := \{x\in\tilde{\mathfrak{g}} \mid [h,x] = \gamma(h)x\}$

$Q_+ = \sum_{i\in I} \mathbb{N}\alpha_i \subset Q$

$\dim \tilde{\mathfrak{g}}_\alpha < \infty \quad \forall\, \alpha\in Q\setminus\{0\}$.

(iii) There is an involution $\tilde{\omega}$ on $\tilde{\mathfrak{g}}$ defined by $e_i \to -f_i$ $f_i \to -e_i$ $h \to -h$

(iv) if $\mathfrak{i} \subset \tilde{\mathfrak{g}}$ is an ideal then $\mathfrak{i} = \bigoplus_{\alpha\in Q\setminus\{0\}} (\mathfrak{i}\cap\tilde{\mathfrak{g}}_\alpha) \oplus (\mathfrak{i}\cap\mathfrak{h})$.

Let $\mathfrak{r}$ = unique maximal ideal of $\tilde{\mathfrak{g}}$ not intersecting $\mathfrak{h}$.

Finally Kac-Moody algebra associated to A is defined to be

$$\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{r}.$$

<u>Remark</u> We will see later that $\mathfrak{r}$ is generated by

$$\mathrm{ad}(e_i)^{1-a_{ij}} e_j \qquad (\mathrm{ad} f_i)^{1-a_{ij}} f_j \qquad \forall\, i\neq j \in I.$$

Again we have $\mathfrak{g}(A) = n_- \oplus \mathfrak{h} \oplus n_+ = \left(\bigoplus_{\alpha\in R_+} \mathfrak{g}_{-\alpha}\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in R_+} \mathfrak{g}_\alpha\right)$

$R_+ \subset Q_+\setminus\{0\}$ set of positive roots.

(1.2) <u>Lemma</u>. If $a \in \mathfrak{n}_+$ is such that $[f_i, a] = 0 \ \forall i \in I$, then $a = 0$. Similarly for $\mathfrak{n}_-$. Center of $\mathfrak{g}(A)$ is given by

$$\mathcal{Z}(\mathfrak{g}(A)) = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0 \ \forall i \in I\} \quad \text{of dim. } |I| - \mathrm{rk}(A).$$

In particular $\mathfrak{g}(A)$ has no non-trivial ideals if and only if $\det A \neq 0$.

<u>Proof</u>. Let i = span of elements of the form $(\mathrm{ad}\, e_i)^k (\mathrm{ad}\, h)^\ell \cdot a$

$(i \in I, \ k, \ell \geq 0, \ h \in \mathfrak{h})$.

Then $i \subset \mathfrak{n}_+$ is an ideal. This contradicts the definition of $\mathfrak{g}(A)$.

For $\alpha \in Q_\pm$ define $ht(\alpha) = \sum_{i \in I} n_i \in \mathbb{Z}$ if $\alpha = \sum_{i \in I} n_i \alpha_i$.

Let $c \in \mathfrak{g}(A)$ be in the center. $c = \sum_{k \in \mathbb{Z}} c_k$. $[f_i, c] = 0 \ \forall i$

$\Rightarrow c_k = 0 \ \forall k > 0$ and similarly for $k < 0$. Thus $c = c_0 \in \mathfrak{h}$.

$0 = [c, e_i] = \alpha_i(c) e_i \Rightarrow \alpha_i(c) = 0 \ \forall i$. Last part follows since ideals are graded. $\square$

(1.3) Bilinear form. Let $\mathfrak{h}' =$ span of h_i's $\subset \mathfrak{h}$. Pick a complementary subspace $\mathfrak{h}''$ and define $(\ ,\ ) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ by

$$(h_i, h) = d_i^{-1} \alpha_i(h) \qquad (h_1, h_2) = 0 \quad \forall h_1, h_2 \in \mathfrak{h}''.$$

Check: kernel of $(\cdot,\cdot)$ restricted to $\mathfrak{h}' = \mathcal{Z}(\mathfrak{g}(A))$. Hence $(\cdot,\cdot)$ is non-degenerate. $\nu : \mathfrak{h} \to \mathfrak{h}^*$ iso. induced by $(\cdot,\cdot)$

We extend $(\cdot,\cdot)$ to $\tilde{\mathfrak{g}}$ by $(e_i, f_j) = \delta_{ij} d_i^{-1}$ and $([x,y],z) = (x,[y,z])$

radical of $(\cdot,\cdot) \subset \mathfrak{r}$ (since $(\cdot,\cdot)|_{\mathfrak{h}}$ is non. deg.)

$\Rightarrow (\cdot,\cdot)$ descends to a symmetric, non-degenerate invariant bilinear form on $\mathfrak{g}$.

(1.4) Weyl group. For each $i \in I$ define $s_i \in \mathrm{Aut}(\mathfrak{h}^*)$ by

$$s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$$

$W \subset GL(\mathfrak{h}^*)$ be the group generated by reflections $\{s_i \mid i \in I\}$. It also acts on $\mathfrak{h}$ through the iso. $\nu : \mathfrak{h} \to \mathfrak{h}^*$. $s_i(h) = h - \alpha_i(h) h_i \quad \forall h \in \mathfrak{h}$.

Note: W preserves the lattice Q and hence is discrete. W also preserves $(\cdot,\cdot)$ on $\mathfrak{h}$ and $\mathfrak{h}^*$.

Partial order on $\mathfrak{h}^*$: $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$.

(1.5) sl_2 representations.

sl_2 = Lie algebra of 2×2 traceless matrices. It has basis given by

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[e, f^k] = -k(k-1) f^{k-1} + k f^{k-1} h.$$

For each $n \in \mathbb{N}$, there is a unique irr. sl_2 representation $L(n)$ with basis $v_0, \ldots, v_n$ and

$$e v_i = (n-i+1) v_{i-1} \qquad f v_i = (i+1) v_{i+1}$$
$$h v_i = (n-2i) v_i$$

(1.6) The following elements are in $\mathfrak{r}$

$$(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j \qquad (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j$$

- $(\mathrm{ad}\, e_i) f_j = 0$
  $(\mathrm{ad}\, h_i) f_j = -a_{ij} f_j$

$$\Rightarrow \quad (\mathrm{ad}\, e_i) \cdot (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j$$
$$= \left[-(1-a_{ij})(-a_{ij}) + (1-a_{ij})(-a_{ij})\right] (\mathrm{ad}\, f_i)^{-a_{ij}} f_j$$
$$= 0$$

Similarly one can show that $[e_k, (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j] = 0$. Hence $(\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0$

Hence adjoint representation is locally nilpotent/ integrable.

(1.7) Let V be a repn. of $\mathfrak{g}$ which is $\mathfrak{h}$-diagonalizable i.e.

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \qquad V_\lambda = \{v \mid h.v = \lambda(h) v \ \forall h \in \mathfrak{h}\} \quad \lambda\text{-weight space.}$$

(assume $\dim V_\lambda < \infty$)!

We say V is <u>integrable</u> if $\forall i \in I$, e_i & f_i are locally nilpotent.

(i.e. $\forall v \in V$. $\exists N > 0$ s.t. $e_i^N v = 0$).

(An extension of) W acts on an integrable representation via

$$\tilde{s}_i = \exp(e_i) \exp(-f_i) \exp(e_i) \qquad \tilde{w} V_\lambda = V_{w\lambda}.$$

and hence also on $\mathfrak{g}$ via $(x \mapsto \tilde{s}_i x \tilde{s}_i^{-1})$. In particular we obtain

Prop. (1) The root system is W invariant. Moreover $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{w\alpha}$ $\forall \alpha \in R$, $w \in W$.

(2) if $\alpha \in R_+$ and $s_i(\alpha) < 0$ then $\alpha = \alpha_i$.

(3)* Let V be an integrable $\mathfrak{g}$-module. $\lambda \in \mathfrak{h}^*$ a weight of V, $i \in I$ (assume $\dim V_\lambda < \infty$) Then

$M := \{t \in \mathbb{Z} \mid \lambda + t\alpha_i$ is a weight of $V\}$.

- $M = [-p, q]$, $p, q \in \mathbb{N}$. $p - q = \lambda(h_i)$.
- $e_i : V_{\lambda + t\alpha_i} \to V_{\lambda + (t+1)\alpha_i}$ is injective for $t < -\frac{1}{2}\lambda(h_i)$

(1.8) Structure of Weyl group.

Lemma. Let $w = s_{i_1} \cdots s_{i_t} \in W$ be a reduced expression.

(i) $\ell(w s_i) < \ell(w) \iff w(\alpha_i) < 0$

(ii) $w(\alpha_{i_t}) < 0$

(iii) If $\ell(w s_i) < \ell(w)$ then $\exists\ 1 \le j \le t$ s.t.

$$s_{i_j} s_{i_{j+1}} \cdots s_{i_t} = s_{i_{j+1}} \cdots s_{i_t} s_i$$

Hence W is a Coxeter group : $W = \langle s_i \mid s_i^2 = 1,\ \underbrace{s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i \cdots}_{m_{ij}} \rangle$

where $m_{ij} = 2, 3, 4, 6, \infty$ if $a_{ij} a_{ji} = 0, 1, 2, 3, \ge 4$ resp.

Proof. $w(\alpha_i) < 0 \Rightarrow \exists j$ s.t. $s_{i_j} \cdots s_{i_t} = s_{i_{j+1}} \cdots s_{i_t} s_i$:

Let $\beta_k = s_{i_{k+1}} \cdots s_{i_t}(\alpha_i)$. $\beta_t > 0$ and $\beta_0 < 0 \Rightarrow \exists j$ s.t. $\beta_{j-1} < 0$ & $\beta_j > 0$. $\beta_{j-1} = s_{i_j}(\beta_j)$. Hence $\beta_j = \alpha_{i_j}$ and we get $\alpha_{i_j} = w(\alpha_i)$ where $w = s_{i_{j+1}} \cdots s_{i_t}$. Therefore $s_{i_j} = w s_i w^{-1}$.

This proves that $w(\alpha_i) < 0 \Rightarrow \ell(ws_i) < \ell(w)$. Conversely if $w(\alpha_i) > 0$ we get $w s_i(\alpha_i) < 0$ and hence $\ell(w s_i s_i) < \ell(w s_i)$. $\square$

(1.9) $C = \{ h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_i(h) \geq 0 \ \forall i \in I \}$ fundamental chamber

$$X = \bigcup_{w \in W} w(C)$$

Prop. (a) Let $h \in C$; $W_h = \{ w \in W \mid w(h) = h \}$. Then W_h is generated by the fundamental reflections $s_i \in W_h$.

(b) C is the fundamental domain for action of W on X.

(c) $X = \{ h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha(h) < 0$ only for a finite number of $\alpha \in R_+ \}$

(d) $C = \{ h \in \mathfrak{h}_{\mathbb{R}} \mid \forall w \in W \quad h - w(h) = \sum_{i \in I} c_i h_i$ with $c_i \geq 0 \}$

(e) TFAE (i) $|W| < \infty$ (ii) $X = \mathfrak{h}_{\mathbb{R}}$ (iii) $|R| < \infty$.

Proof. (a) and (b) by induction on $\ell(w)$ as follows. Let $h \in C$ and $w(h) = h' \in C$. $w = s_{i_1} \cdots s_{i_\ell}$ reduced expression for w. Then $\alpha_{i_\ell}(h) \geq 0$ and $(w\alpha_{i_\ell})(h') \geq 0$ but $w\alpha_{i_\ell} < 0 \Rightarrow \alpha_{i_\ell}(h) = 0 = w\alpha_{i_\ell}(h')$

So $s_{i_\ell}(h) = h$.

(c) Let $X' = \{ h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha(h) < 0$ only for a finite number of $\alpha \in R_+ \}$

Then $C \subset X'$ and X' is W-invariant $\Rightarrow X \subset X'$

Conversely let $h \in X'$ and let $M_h = \{ \alpha \mid \alpha(h) < 0 \}$. We argue by induction on $|M_h|$ since $\alpha_i \in M_h$ for some i and hence $|M_{s_i h}| < |M_h|$

(d) $\supset$ is clear. We prove the converse by induction. Let $h \in \mathfrak{h}_{\mathbb{R}}$ be an element of C and $w = s_{i_1} \dots s_{i_\ell}$ a reduced expression. If $\ell = 1$, $h - w(h) \in \sum \mathbb{R}_{\geq 0} h_i$ by definition. Otherwise

$$h - w(h) = (h - s_{i_1} \dots s_{i_{\ell-1}} h) + s_{i_1} \dots s_{i_{\ell-1}} (h - s_{i_\ell}(h))$$

(e): (i) $\Rightarrow$ (ii) Let $h \in \mathfrak{h}_{\mathbb{R}}$. Pick the maximal element of Wh. It must lie in C. (ii) $\Rightarrow$ (iii) Let $h \in$ interior of C, then $\alpha(-h) < 0 \ \forall \alpha \in R_+$. $-h \in X \Rightarrow |R_+| < \infty$. (iii) $\Rightarrow$ (i) we claim that $w(\alpha) = \alpha \ \forall \alpha \in R$ implies $w = 1$. Hence $W \subset$ Permutation group of R_+. The proof is easy consequence of the exchange property (Lemma (1.8) (iii)).

(1.10) The numbers m_{ij}. $|I| = 2$ $\quad a_{12} = -a \quad a_{21} = -b$

$$s_1 = \begin{bmatrix} -1 & -a \\ 0 & 1 \end{bmatrix} \qquad s_2 = \begin{bmatrix} 1 & 0 \\ -b & -1 \end{bmatrix} \Rightarrow s_1 s_2 = \begin{bmatrix} -1+ab & a \\ -b & -1 \end{bmatrix}$$

$$\det(\lambda I - s_1 s_2) = (\lambda + 1 - ab)(\lambda + 1) + ab = \lambda^2 + (2 - ab)\lambda + 1$$

$s_1 s_2$ is of finite order m iff $ab < 3$ with $m = 2, 3, 4, 6$ if $ab = 0, 1, 2, 3$ resp.

(1.11) Classification of Cartan matrices.

Let us work in more generality. $A \in M_{I \times I}(\mathbb{R})$ indec., symmetrizable (DA is symmetric, $D = (d_i)_{i \in I}$, $d_i \geq 0$) and $a_{ij} \leq 0 \ \forall i \neq j$.

We write $u > 0$ for a real column vector u if all its entries are > 0.

Similarly $u \geq 0$ if $u_i \geq 0$.

Thm. A is one of the following

(Fin) $\det A \neq 0$. $\exists\, u > 0$ s.t. $Au > 0$. $\quad Au \geq 0 \Rightarrow Au > 0$ or $u = 0$.

(Aff) $\mathrm{rk}(A) = |I| - 1$. $\exists\, u > 0$ s.t. $Au = 0$. $\quad Au \geq 0$ ~~$u \geq 0$~~ $\Rightarrow Au = 0$

(Ind) $\exists\, u > 0$ s.t. $Au < 0$. $\quad Au \geq 0$ & $u \geq 0 \Rightarrow u = 0$

In particular, A is of finite, affine or indefinite types $\Leftrightarrow \exists\, \alpha > 0$ s.t. $A\alpha > 0$, $A\alpha = 0$, $A\alpha < 0$ resp. A is finite or affine iff DA is positive definite (positive semi-definite)

Proof. Let us symmetrize A, so it suffices to work with the symmetric case.

$K_A := \{u \mid Au \geq 0\}$ $\qquad \mathcal{C} := \{u \geq 0\}$. Consider the intersection $K_A \cap \mathcal{C}$. Indecomposability of A implies that $Au \geq 0$ & $u \geq 0 \Rightarrow u > 0$ or $u = 0$

i.e. K_A meets boundry of $\mathcal{C}$ exactly at 0. There are three possibilities

(i) $K_A \subset \{u > 0\} \cup \{0\}$

(ii) K_A = Kernel of A is 1-dim'l subspace spanned by $u > 0$

(iii) $K_A \cap \mathcal{C} = \{0\}$.

(i) is equivalent to (Fin). (ii) is equivalent to (Aff) and (iii) is equivalent to (Ind). For the last part (of (Ind)) we have to use the fact that a system of linear inequalities $\{\lambda_1 > 0 \,..\, \lambda_p > 0\}$ has a soln if and only if there is no linear reln b/w $\lambda_1 \,..\, \lambda_p$ with non-negative coefficients. (non-trivial)

Now one can classify all finite type Cartan matrices (see below).

(1.12) Thm. TFAE : (1) $\mathfrak{g}$ is f.d. (2) A is of finite type (3) $|W| < \infty$ (4) $|R| < \infty$.

Proof. (2) $\Rightarrow$ (3) W is a discrete subgroup of $O(\mathfrak{h}_{\mathbb{R}}, (\cdot,\cdot))$ positive definite

(3) $\Leftrightarrow$ (4) earlier (4) $\Rightarrow$ (1) clear

(1) $\Rightarrow$ (2) $\mathfrak{g}$ finite dim'l $\Rightarrow \exists\, \alpha \in R_+$ s.t. $\alpha + \alpha_i \notin R_+$ $(\forall i)$. Hence $\alpha(h_i) \geq 0$ $(\forall i)$ So $A\alpha \geq 0 \Rightarrow A$ is either finite or affine.

In the affine case $\alpha(h_i) = 0$ $(\forall i)$. Moreover $\exists\, i$ s.t. $\alpha - \alpha_i \in R_+$

By sl_2-repn theory as $\{e_i, f_i, h_i\}$-module $\mathfrak{g}_\alpha \neq 0$ (0-weight space)

$\mathfrak{g}_{\alpha-\alpha_i} \neq 0$

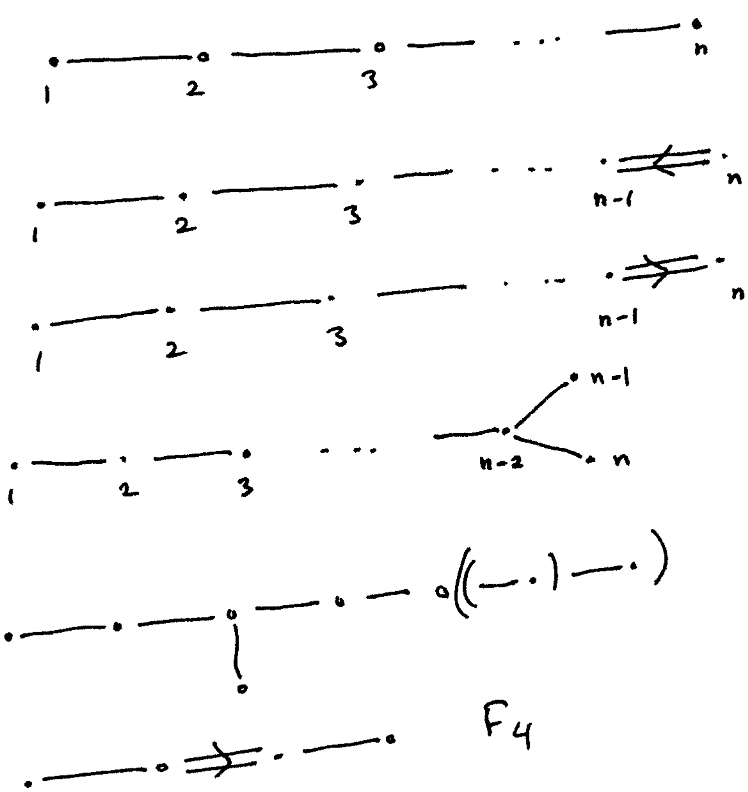
$\Rightarrow \mathfrak{g}_{\alpha+\alpha_i} \neq 0$ which is contradiction $\square$

(1.13) List of finite type Cartan matrices.

Dynkin diagram is a graph on vertex set I

$a_{ij} = a_{ji} = -1$ $\qquad$ $a_{ij} = -1 \quad a_{ji} = -2$ $\qquad$ $a_{ij} = -1 \quad a_{ji} = -3$

A is of finite type iff its Dynkin diagram is one of the following



A_n

B_n

C_n

D_n

$E_{6,7,8}$

G_2

F_4

(1.14) For rest of the course $\mathfrak{g}$ = f.d. simple Lie algebra

$\mathfrak{h}$ = Cartan subalgebra

$A = (a_{ij})_{i,j \in I}$ Cartan matrix

$(\cdot,\cdot)$ invariant bilinear form on $\mathfrak{g}$

$D = (d_i : i \in I)$ symmetrizing matrix

Q = root lattice

$R = R_- \cup R_+ \subset \mathfrak{h}^* \setminus \{0\}$ root system

$<$ partial order on $\mathfrak{h}^*$

$\Delta = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ simple roots

$\nu : \mathfrak{h} \to \mathfrak{h}^*$ iso.

$W = \langle s_i \mid s_i^2 = 1 \quad s_i s_j \cdots = s_j s_i \cdots \rangle$ Weyl group

$\theta \in R_+$ highest root

$\rho \in \mathfrak{h}^*$ s.t. $\rho(h_i) = 1$ $(\forall i \in I)$