Lecture 10

(10.0) Recall that we have been studying the category of finite dimensional representations of $\mathcal{U}_q(L\mathfrak{g})$. We constructed a few examples that indicate the contrast between $\mathcal{O}_{int}$ and $\mathrm{Rep}_{fd}(\mathcal{U}_q(L\mathfrak{g}))$, mainly:

(1) $\mathcal{O}_{int}$ is semisimple, $\mathrm{Rep}_{fd}(\mathcal{U})$ is not.

(2) The set of isomorphism classes of simple objects in $\mathrm{Rep}_{fd}(\mathcal{U})$ is in bijection with a continuous set $= \mathbb{C}_0[w]^I$, where $\mathbb{C}_0[w] \subset \mathbb{C}[w]$ consists of monic polynomials with non-zero constant term

(3) $\mathrm{Rep}_{fd}(\mathcal{U})$ is not braided, i.e. $V \otimes W$ is not necessarily isomorphic to $W \otimes V$.

Aim of today's lecture is to construct a meromorphic braiding on $\mathrm{Rep}_{fd}(\mathcal{U})$:

Theorem. For $V, W \in \mathrm{Rep}_{fd}(\mathcal{U})$, there exists a meromorphic function ($\mathrm{End}(V \otimes W)$ - valued) of $\zeta \in \mathbb{C}^\times$, denoted by $\mathcal{R}_{V,W}(\zeta)$ s.t.

$$(12) \circ \mathcal{R}_{VW}(\zeta) : V(\zeta) \otimes W \longrightarrow W \otimes V(\zeta)$$

is a morphism in $\mathrm{Rep}_{fd}(\mathcal{U})$. In particular, for generic ζ,

$$V(\zeta) \otimes W \xrightarrow{\sim} W \otimes V(\zeta).$$

(10.1) Construction of $\mathcal{R}_{VW}(\zeta)$.

Recall that we have $\tilde{\mathcal{R}} \in \mathcal{U}_q(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q(\hat{\mathfrak{g}})$ (completed tensor) which makes $\mathcal{U}_q\hat{\mathfrak{g}}$ a quasi-triangular Hopf algebra, i.e.

$$\Delta^{op}(x) = \tilde{\mathcal{R}}\, \Delta(x)\, \tilde{\mathcal{R}}^{-1} \qquad \forall\ x \in \mathcal{U}_q(\hat{\mathfrak{g}})$$

$$\Delta \otimes 1\, (\tilde{\mathcal{R}}) = \tilde{\mathcal{R}}_{13}\, \tilde{\mathcal{R}}_{23}$$

$$1 \otimes \Delta\, (\tilde{\mathcal{R}}) = \tilde{\mathcal{R}}_{13}\, \tilde{\mathcal{R}}_{12}$$

$\tilde{R}$ has the following form:

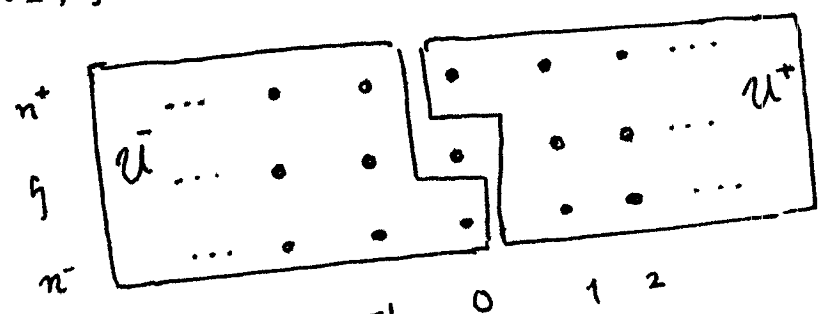
$$\tilde{R} = q^{m(c\otimes d + d\otimes c)} \, q^{\sum x_a \otimes x_a} \cdot \sum_t u_t \otimes u^t \qquad \text{where}$$

$m = 1, 2$ or 3 if $\mathfrak{g}$ is of type ADE, BCF or G.

$\{x_a\}$ is orthonormal basis of $\mathfrak{h}$. $\{u_t\}$ is a homogeneous basis of $\mathcal{U}^+$ and $\{u^t\}$ is dual basis of $\mathcal{U}^-$

($\mathcal{U}^+$ = subalg. gen. by $e_i \; (i \in \hat{I})$; $\mathcal{U}^-$ = subalg. gen. by $f_i \; (i \in \hat{I})$)

$$\Omega^0 := \sum x_a \otimes x_a$$



Define

$$\tilde{R}(\zeta) = \tau_\zeta \otimes 1 \, (R) = q^{m(c\otimes d + d \otimes c)} \, q^{-2\Omega} \cdot \bar{R}(\zeta)$$

$$\bar{R}(\zeta) \in \mathcal{U}^+ \otimes \mathcal{U}^- [\![\zeta]\!]$$

c acts as 0 on finite-dimensional representations, thus the factor $m(c\otimes d + d\otimes c)$ is irrelevant on $V \otimes W$. Set

$$R(\zeta) = q^{\Omega} \, \bar{R}(\zeta) \qquad R_{VW}(\zeta) = (\pi_V \otimes \pi_W)(R(\zeta)) \in \mathrm{End}(V \otimes W) [\![\zeta]\!]$$

Note: $R(0) = R$- matrix of $\mathcal{U}_q(\mathfrak{g})$.

Clearly $R_{VW}(\zeta)$ satisfies all the properties stated in the main theorem, formally. More precisely, let us define

$$\Delta_\zeta := \tau_\zeta \otimes 1 \circ \Delta : \mathcal{U} \longrightarrow \mathcal{U} \otimes \mathcal{U} [\zeta, \zeta^{-1}]$$

Then we have

(1) $\quad R(\zeta)\,\Delta_\zeta(x) = \Delta^{op}_\zeta(x)\cdot R(\zeta) \quad \forall\, x \in \mathcal{U}$

(2) $\quad \Delta\otimes 1 \; R(\zeta) = R_{13}(\zeta)\, R_{23}(\zeta)$

$\quad\;\; 1\otimes\Delta \; R(\zeta) = R_{13}(\zeta)\, R_{12}(\zeta)$

(3) $\quad \tau_\alpha \otimes \tau_\beta \; R(\zeta) = R(\zeta\alpha\beta^{-1})$

(10.2) Convergence of $R_{VW}(\zeta)$.

- Chari and Pressley computed $R_{VW}(\zeta)$ for $\mathfrak{g} = sl_2$ and V, W are irreducible representations. Actually they explicitly wrote down the intertwiner $V(\zeta)\otimes W \to W\otimes V(\zeta)$ normalized so that it maps highest weight vector $v\otimes w$ to $w\otimes v$. By irreducibility, $R_{VW}(\zeta) = f_{VW}(\zeta)(\sigma\circ\text{Interlwiner})$ where $f_{VW}(\zeta) \in \mathbb{C}[\![\zeta]\!]$. They did not prove convergence of this $f_{VW}(\zeta)$.
- Frenkel Etingof and Kirillov proved convergence of this scalar series for arbitrary $\mathfrak{g}$
- Kazhdan-Soibelman proved meromorphicity of $R_{VW}(\zeta)$ for arbitrary V and W; and arbitrary $\mathfrak{g}$ using ideas of [EFK].

Main Idea: (1) R_{VW} satisfies a q-difference equation, called crossing symmetry.

(2) Formal solns of q-difference equations converge.

The proof given below is due to Etingof and Moura (of part (2))

Crossing symmetry was obtained by Etingof-Frenkel-Kirillov.

(10.3) Crossing Symmetry.

Lemma. Let $V \in \mathrm{Rep}_{fd}(\mathcal{U})$. Then we have natural iso.

$$V(\zeta)^{**} \simeq V(q^{2mh^\vee}\zeta) \quad \text{given by} \quad q^{2\rho}. \qquad (h^\vee = 1+\rho(\theta^\vee))$$

Proof. Recall $S^2(x) = q^{2\hat\rho}\, x\, q^{-2\hat\rho}$ where $\hat\rho \in \tilde{\mathfrak{h}}^*$ is such that $(\hat\rho, \alpha_i) = d_i \ \forall i \in \hat{I}$. Let us take $\hat\rho = \rho + h^\vee \Lambda_0$ (alternately we can say $\hat\rho \leftrightarrow mh^\vee d + \nu^{-1}(\rho) \in \tilde{\mathfrak{h}}$). Hence

$S^2(x) = q^{2\rho}\, \tau_{q^{2mh^\vee}}(x)\, q^{-2\rho}$; and action of $q^{2\rho}$ is a morphism

$$V(q^{2mh^\vee}\zeta) \xrightarrow{q^{2\rho}} V(\zeta)^{**}$$

□

Notation: for a linear operator $A : V\otimes W \to V \otimes W$ define A^{t_1} to be the operator it defines $V^*\otimes W \to V^*\otimes W$.

Prop.

$$R_{VW}(q^{2mh^\vee}\zeta) = q^{-2\rho}\otimes 1 \left(\left(\left(R_{VW}(\zeta)^{-1}\right)^{t_1}\right)^{-1}\right)^{t_1} q^{2\rho}\otimes 1$$

Proof. Compute $R_{V^{**},W}(\zeta)$ in two different ways

- Using the Lemma: $\mathrm{Ad}(q^{2\rho}\otimes 1)\, R_{VW}(p\cdot\zeta) \qquad p = q^{2mh^\vee}$
- using the identity $S\otimes 1(R) = R^{-1}$

$$\equiv \quad R_{V^*,W}(\zeta) = \left(R_{VW}(\zeta)^{-1}\right)^{t_1}$$

□

Hence $R_{VW}(\zeta)$ formally satisfies a q-difference equation

$R(p\cdot\zeta) = G(R(\zeta))$ where G G $\mathrm{End}(V\otimes W)$ by

$$G(X) = \mathrm{Ad}(q^{2\rho}\otimes 1)\left(\left(\left(X^{-1}\right)^{t_1}\right)^{-1}\right)^{t_1}$$

(10.4) Formal solutions converge.

Let $\mathcal{E} = \mathbb{C}^N$ and U be an open nhd. of $0 \in \mathcal{E}$. Let $G : U \to \mathcal{E}$ be an analytic function s.t. $G(0) = 0$. Let $p \in \mathbb{C}$ be of modulus > 1.

(in our case $\mathcal{E} = \mathrm{End}(V \otimes W)$, $G(X) = (((X^{-1})^{t_1})^{-1})^{t}$ conjugated by $q^{2\rho} \otimes 1$.
We shift the vector space $\mathcal{E}$ to make $R =$ origin for convenience.
Further $p = q^{2mh^\vee}$ and we assume $|q| > 1$)

Choose local coordinates $z_1 \dots z_N$ for the open set U, so that G can be written as a formal series

$$G = \sum_{\alpha \in \mathbb{N}^N \setminus \underline{0}} a_\alpha \underline{z}^\alpha \qquad (\underline{z}^\alpha = z_1^{\alpha_1} \dots z_N^{\alpha_N}) \quad a_\alpha \in \mathcal{E}$$

Let $f(z) \in \mathcal{E}[\![z]\!]$, $f(0) = 0$ be a formal solution of

$$f(pz) = G(f(z))$$

(if we write $f(z) = \sum_{i=1}^N \epsilon_i f_i(z)$, ϵ_i basis of $\mathcal{E}$ dual to z_i's then

$$G(f(z)) = \sum_{\alpha \in \mathbb{N}^N \setminus \underline{0}} a_\alpha f_1(z)^{\alpha_1} \dots f_N(z)^{\alpha_N} \in \mathcal{E}[\![z]\!] \ )$$

Prop. $f(z)$ converges to an analytic function in some neighborhood of 0.

Proof. Define $g_m : S^m \mathcal{E} \to \mathcal{E}$ by

$$g_m(\epsilon_1^{t_1} \dots \epsilon_N^{t_N}) = a_{t_1 \dots t_N} \qquad (t_1 + \dots + t_N = m;\ t_i \geq 0)$$

Convergence of $G \Rightarrow \exists A > 0$ s.t. $\|g_m\| < A^m$.

Note $$G(f(z)) = \sum_{m \geq 1} \Big(\sum_{t_1 + \dots + t_N = m} g_m(\epsilon_1^{t_1} \dots \epsilon_N^{t_N}) f_1(z)^{t_1} \dots f_N(z)^{t_N} \Big)$$

Write $$f(z) = \sum_{i \geq 1} f[i] z^i \qquad f[i] = \epsilon_1 f_{1;i} + \dots + \epsilon_N f_{N;i} \in \mathcal{E}$$

Then coefficient of z^l in $G(f(z))$ equals

$$\sum_{r\geq 1} \sum_{\substack{i_1+\dots+i_r=l \\ i_1,\dots,i_r\geq 1}} g_r\left(f[i_1]\cdots f[i_r]\right)$$

- Let $k_0 \in \mathbb{N}$ be large enough so that $\forall\, k \geq k_0$ we have
$\boxed{|(p^k - g_1)^{-1}| < 2|p|^{-k}}^{*}$ and $2A < |p|^{k/2}$.
- $C > 0$ be s.t. $|f_k| < C \;\; \forall\, k < k_0$ and $B > \max\left(1, \frac{AC}{|p|^{1/2}-1}\right)$.

Claim: $\quad |f[k]| < C\, B^{k-1}$.

The claim holds for $k < k_0$. Assume it for all $n < k$ $(k \geq k_0)$

Comparing coeff of z^k in $f(pz) = G(f(z))$ we get

$$p^k f[k] = g_1(f[k]) + \sum_{r>1} \sum_{i_1+\dots+i_r=k} g_r\left(f[i_1]\cdots f[i_r]\right)$$

$$\Rightarrow \quad f[k] = (p^k - g_1)^{-1} \sum_{r>1} \sum_{i_1+\dots+i_r=k} g_r\left(f[i_1]\cdots f[i_r]\right)$$

$$|f[k]| < 2|p|^{-k} \sum \binom{k-1}{r-1} (AC)^r B^{k-r} = 2\,(AC/B)\left(1+\frac{AC}{B}\right)^{k-1} |p|^{-k} B^k$$

$$2A < |p|^{k/2} \quad \text{and} \quad 1 + \frac{AC}{B} < p^{1/2}$$

$$\Rightarrow \quad |f[k]| < |p|^{k/2}\, |p|^{\frac{k-1}{2}}\, |p|^{-k}\, C B^{k-1} < C B^{k-1}$$

(10.5) Meromorphic tensor categories. I.

(Recall the definition of braided monoidal categories from Lecture 3)

First definition (Frenkel - Reshetikhin)

Let $\mathcal{C}$ be a $\mathbb{C}$-linear monoidal category and G be a (complex) group acting on $\mathcal{C}$. Assume there is a forgetful functor $F : \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$.

Notation. $g \in G$, $X \in \mathcal{C}$ $\quad g \cdot X =: X(g) \qquad F(X(g)) = F(X)$

Meromorphic associativity constraint :

$\forall\ X_1\ X_2\ X_3 \in \mathcal{C}$ $\qquad a_{X_1 X_2 X_3}(g_1, g_2, g_3) : (X_1(g_1) \otimes X_2(g_2)) \otimes X_3(g_3)$

$g_1\ g_2\ g_3 \in G$ $\qquad \xrightarrow{\sim} X_1(g_1) \otimes (X_2(g_2) \otimes X_3(g_3))$

s.t. $F(a_{X_1 X_2 X_3}(g_1, g_2, g_3)) \in \mathrm{End}\,(F(X_1) \otimes F(X_2) \otimes F(X_3))$ is meromorphic function on $G \times G \times G$. (generically invertible)

$\Rightarrow$ Impose commutativity of all diagrams (e.g. pentagon axiom).

Meromorphic commutativity constraint

$\forall\ X, Y \in \mathcal{C}$ $\qquad c_{XY}(g_1, g_2) : X(g_1) \otimes Y(g_2) \to Y(g_2) \otimes X(g_1)$

$g_1, g_2 \in G$ $\qquad F(c_{X,Y}(g_1, g_2)) \in \mathrm{Hom}\,(X \otimes Y, Y \otimes X)$ meromorphic.

We will drop associativity constraint since the categories we consider are strict.

More sophisticated version is due to Y. Soibelman, building upon works of Beilinson and Drinfeld.