Lecture 11 Meromorphic Tensor Categories

(11.0) Motivation. Let us define another coproduct on $\mathcal{U} = \mathcal{U}_q(L\mathfrak{g})$ as follows.

$$\Delta_\zeta(\psi^\pm_{i,\pm m}) = \sum_{s=0}^{m} \zeta^{\pm(m-s)} \psi^\pm_{i,\pm(m-s)} \otimes \psi^\pm_{\pm s} \qquad \left(\begin{matrix} m\in\mathbb{N},\ k\in\mathbb{Z} \\ i\in I\end{matrix}\right)$$

$$\Delta_\zeta(E_{i,k}) = \zeta^k E_{i,k}\otimes 1 + \sum_{\ell\geq 0} \zeta^{-\ell}\psi^-_{i,-\ell}\otimes E_{i,k+\ell}$$

$$\Delta_\zeta(F_{i,k}) = 1\otimes F_{i,k} + \sum_{\ell\geq 0}\zeta^{k-\ell} F_{i,k-\ell}\otimes\psi^+_{i,\ell}$$

For $\zeta=1$, this coproduct was introduced by Drinfeld, and is called Drinfeld coproduct in his honor. The expression of Δ_ζ given above is due to Hernandez who coined the term deformed Drinfeld coproduct for Δ_ζ and proved that

- $\Delta_\zeta : \mathcal{U} \to \mathcal{U}\otimes\mathcal{U}((\zeta^{-1}))$ is an algebra homomorphism
- Coassociativity $\Delta_{\zeta_1}\otimes 1 \circ \Delta_{\zeta_2} = 1\otimes\Delta_{\zeta_2}\circ\Delta_{\zeta_1\zeta_2}$

Expression of Δ_ζ as a contour integral was obtained only recently by G. and Toledano Laredo. Namely let V and W be f.d. repns. of $\mathcal{U}_q(L\mathfrak{g})$

$\sigma(V)$ = poles of rational functions $\psi_i(z)$, $E_i(z)$, $F_i(z)$ acting on V

(similarly $\sigma(W)$).

$$\Delta_\zeta(\psi_i(z)) = \psi_i(\zeta^{-1}z)\otimes\psi_i(z)$$

$$\Delta_\zeta(E_i(z)) = E_i(\zeta^{-1}z)\otimes 1 + \oint_{C_2}\frac{zw^{-1}}{z-w}\psi_i(\zeta^{-1}w)\otimes E_i(w)\,dw$$

$$\Delta_\zeta(F_i(z)) = \oint_{C_1}\frac{zw^{-1}}{z-\zeta w}F_i(w)\otimes\psi_i(\zeta w)\,dw + 1\otimes F_i(z)$$

- C_1 and C_2 are contours enclosing $\sigma(V)$ and $\sigma(W)$ resp.
- $\Delta_\zeta(E_i(z))$: $\displaystyle\oint_{C_2}\frac{zw^{-1}}{z-w}\psi_i(\zeta^{-1}w)\otimes E_i(w)\,dw$ defines a rational function

of ζ and z in the domain where z is not enclosed by C_2 and $\pi_V(\psi_i(\zeta^{-1}w))$ is analytic (holomorphic) within C_2.

- $\Delta_\zeta(F_i(z))$: $\displaystyle\oint_{C_1} \frac{zw^{-1}}{z-\zeta w} F_i(w) \otimes \psi_i(\zeta w)\, dw$ ← rational function of ζ and z in the domain where z is not enclosed by $\zeta . C_1$ and $\pi_W(\psi_i(\zeta w))$ is holomorphic within C_1.

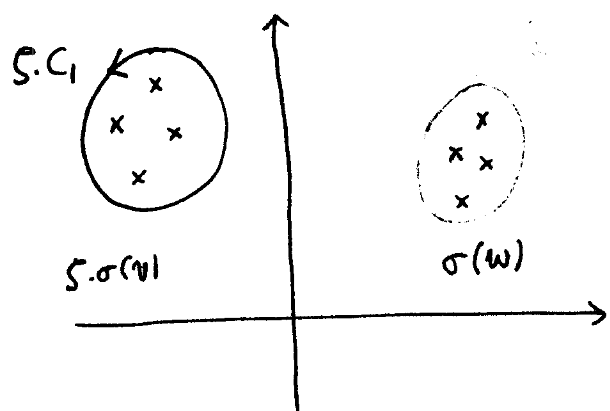
Thus we obtain an algebra hom. $\Delta_\zeta : \mathcal{U} \to \mathrm{End}(V\otimes W)(\zeta)$

Δ_ζ has poles at $\zeta = \beta\alpha^{-1}$ where $\beta \in \sigma(W)$ and $\alpha \in \sigma(V)$.

Alternately, given V and W, pick $\zeta \in \mathbb{C}^\times$ s.t. $\zeta.\sigma(V) \cap \sigma(W) = \phi$ ζC_1 and C_2 are disjoint and define $\mathcal{U} \curvearrowright V\otimes W$ by these expressions.

Notation: $V \underset{\zeta}{\otimes} W$ the resulting representation.

Remark: $V \underset{\zeta}{\otimes} W = V(\zeta) \underset{1}{\otimes} W$



Example. $\mathcal{U}_q(L\mathfrak{sl}_2)$ acts on $L_{m_1}(\alpha)$ and $L_{n_2}(\beta)$ $\quad m,n \in \mathbb{N}$; $\alpha, \beta \in \mathbb{C}^\times$.

(see section (9.2))

$$e(z)\, m_n(r) = \frac{[n-r+1]\, z}{z - \alpha q^{n-2r+1}}\, m_n(r-1) \qquad (0 \le r \le n)$$

$$f(z)\, m_n(r) = \frac{[r+1]\, z}{z - \alpha q^{n-2r-1}}\, m_n(r+1)$$

$$\psi(z)\, m_n(r) = q^{n-2r} \frac{(z - \alpha q^{n+1})(z - \alpha q^{-n-1})}{(z - \alpha q^{n-2r-1})(z - \alpha q^{n-2r+1})}\, m_n(r)$$

$$\sigma(L_n(\alpha)) = \{\alpha q^{-n+1}, \dots, \alpha q^{n-1}\}$$

$$\Delta_\zeta(\psi(z))\, m_{n_1}(r_1) \otimes m_{n_2}(r_2) = q^{n_1+n_2-2r_1-2r_2} \frac{(z - q^{n_1+1}\zeta\alpha_1)(z - q^{-n_1-1}\zeta\alpha_1)}{(z - q^{n_1-2r_1-1}\zeta\alpha_1)(z - q^{n_1-2r_1+1}\zeta\alpha_1)} \frac{(z - q^{n_2+1}\alpha_2)(z - q^{-n_2-1}\alpha_2)}{(z - q^{n_2-2r_2-1}\alpha_2)(z - q^{n_2-2r_2+1}\alpha_2)}$$

$$\cdot\, m_{n_1}(r_1) \otimes m_{n_2}(r_2)$$

$$\Delta_\zeta(e(z))\, m_{n_1}(r_1) \otimes m_{n_2}(r_2) = \frac{[n_1 - r_1 + 1]\, z}{z - \zeta\alpha_1 q^{n_1-2r_1+1}}\, m_{n_1}(r_1 - 1) \otimes m_{n_2}(r_2) \ +$$

$$q^{n_1-2r_1} \frac{[n_2 - r_2 + 1]\, z}{z - \alpha_2 q^{n_2-2r_2+1}} \frac{(q^{n_1+1}\zeta\alpha_1 - q^{n_2-2r_2+1}\alpha_2)(q^{-n_1-1}\zeta\alpha_1 - q^{n_2-2r_2+1}\alpha_2)}{(q^{n_1-2r_1-1}\zeta\alpha_1 - q^{n_2-2r_2+1}\alpha_2)(q^{n_1-2r_1+1}\zeta\alpha_1 - q^{n_2-2r_2+1}\alpha_2)}\, m_{n_1}(r_1) \otimes m_{n_2}(r_2 - 1)$$

This example does not fit in the framework of naive meromorphic categories introduced last time, since we don't have a bifunctor

$$\otimes : \mathrm{Rep}(\mathcal{U}) \times \mathrm{Rep}(\mathcal{U}) \to \mathrm{Rep}(\mathcal{U}).$$

$V \underset{\zeta}{\otimes} W$ only exists for $\zeta \notin \{\beta\alpha^{-1} \mid \beta \in \sigma(W), \alpha \in \sigma(V)\} \leftarrow$ a finite subset of $\mathbb{C}^\times$ (depending on V & W).

Aim: to introduce a more sophisticated notion of meromorphic tensor categories (due to Y. Soibelman), so that

$(\mathrm{Rep}\,\mathcal{U}, \otimes)$ and $(\mathrm{Rep}\,\mathcal{U}, \underset{\zeta}{\otimes})$ are both mero. tensor categories

(also braided).

Braiding on $(\mathrm{Rep}\,\mathcal{U}, \otimes)$ was constructed last time. On $(\mathrm{Rep}\,\mathcal{U}, \underset{D}{\otimes})$ we will give a mero. braiding in future.

(11.1) Schematic diagram for Soibelman's definitions

Pseudo monoidal category   Pseudo braided category

$\Downarrow$ (representability)

Monoidal (resp. braided monoidal) categories

Introduce spaces (+ symmetric group action)

Pseudo monoidal (resp. braided monoidal) categories / a space

Meromorphic $\longleftrightarrow$ spaces are irr. algebraic varieties / $\mathbb{C}$
(smooth)

Special case : Categories equipped with G-action
"Spaces" are defined using G

we get pseudo meromorphic G-tensor (or braided tensor) categories

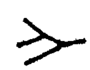
Then we drop the adjective pseudo if representability holds.

(11.2) Let T_n = set of complete bracketings on n letters
= set of planar binary trees (n incoming and 1 outgoing edge)

Concatenation operation

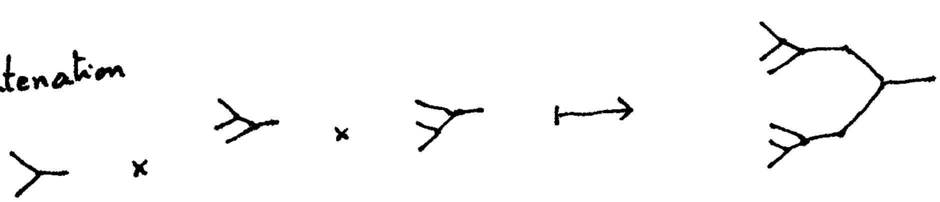
$$T_n \times T_{k_1} \times \cdots \times T_{k_n} \longrightarrow T_{k_1 + \cdots + k_n}$$

e.g. $n = 3$

$(\bullet\,\bullet)\,\bullet \longleftrightarrow$ 

$\bullet\,(\bullet\,\bullet) \longleftrightarrow$ 

Concatenation



$$(T_2 \times T_3 \times T_3 \longrightarrow T_6)$$

Concatenation is often denoted by

$$(b, b_1, \dots, b_n) \longmapsto b(b_1 \cdots b_n)$$

Definition. Pseudo monoidal category [Beilinson-Drinfeld]

Let $\mathcal{C}$ be a class of objects. Assume the following data is given

(1) $\forall\ b \in T_n$, $X_1, \dots, X_n, Y \in \mathcal{C}$ we have a $\mathbb{C}$-vector space

$$P_b(X_1, \dots, X_n; Y)$$

(2) Composition: $\forall\ b \in T_n$ $b_1 \cdots b_n \in T_{k_1} \times \dots \times T_{k_n}$

$X_1, \dots, X_n$, $\{Y_{i,1} \dots Y_{i,k_i}\}_{i=1\dots n}$, Z objects from $\mathcal{C}$

$$P_b(X_1 \dots X_n; Z) \times \prod_{i=1}^{n} P_{b_i}(Y_{i_1} \dots Y_{i,k_i}; X_i) \to P_{b(b_1 \cdots b_n)}(\{Y_{i,j}\}_{\substack{i=1\dots n \\ j=1\dots k_i}}; Z)$$

(3) A distinguished element $\mathrm{Id}_X \in P_{\rightarrow}(X; X)$

(4) Associator: a (natural) iso. of vector spaces $\forall\ b, b' \in T_n$ $P_b(\underline{X}, Y) \overset{\wr}{\downarrow}{a_{b',b}}\, P_{b'}(\underline{X}, Y)$

$$P_{\succ}(X_1, X_2, X_3; Y) \xrightarrow{\sim} P_{\succ}(X_1, X_2, X_3; Y)$$

Satisfying the following axioms:

(1) Composition is associative

For every $b \in T_n$ $b_1 \dots b_n$ $(b_i \in T_{k_i})$ $b_{i,1} \dots b_{i,k_i}$ $(1 \le i \le n)$ $b_{i,r} \in T_{l_{i,r}}$ $(1 \le r \le k_i)$

objects $X_1 \dots X_n$ $Y_{i,1} \dots Y_{i,k_i}$ $(1 \le i \le n)$ $Z_{i,r,1} \dots Z_{i,r,l_{i,r}}$ $(1 \le r \le k_i$ and $1 \le i \le n)$ and U

The following diagram is commutative

$$\begin{tikzcd}[column sep=tiny]
& P_b(\underline{X}, U) \times \prod\limits_{i=1\dots n} P_{b_i}(\underline{Y}_i, X_i) \times \prod\limits_{\substack{r=1\dots k_i\\ i=1\dots n}} P_{b_{i,r}}(\underline{Z}_{i,r}, Y_{i,r}) \arrow[dl] \arrow[dr] & \\
P_{b(b_1\dots b_n)}(\underline{Y}, U) \times \prod\limits_{\substack{r=1\dots k_i\\ i=1\dots n}} P_{b_{i,r}}(\underline{Z}_{i,r}, Y_{i,r}) \arrow[dr] & & P_b(\underline{X}, U) \times \prod\limits_{i=1}^{n} P_{b_i(b_{i,1}\dots b_{i,k_i})}(\underline{Z}_i, X_i) \arrow[dl] \\
& P_{b(b_1\dots b_n)(b_{1,1}\dots b_{n,k_n})}(\underline{Z}, U) &
\end{tikzcd}$$

For notational convenience we abbreviated, for instance

$$\underline{Z}_{i,r} = Z_{i,r,1} \dots Z_{i,r,l_{i,r}}$$ and so on.

(2) Composition with Id_X is identity

$$P_b(X_1\dots X_n;Y) \times P_{\to}(Y,Y) \longrightarrow P_b(X_1\dots X_n;Y)$$ and similarly.

$$(\psi, \mathrm{Id}_Y) \longmapsto \psi$$

$$P_{\to}(X_i, X_i) \times P_b(X_1\dots X_n;Y) \longrightarrow P_b(X_1\dots X_n;Y)$$

(3) Pentagon axiom.

$$a_{b''b'}\, a_{b'b} = a_{b''b}$$

$a_{b',b}$ commutes with composition

$$\begin{tikzcd}[column sep=small]
& (\bullet\,\bullet)(\bullet\,\bullet) \arrow[dr] & \\
((\bullet\,\bullet)\,\bullet)\,\bullet \arrow[ur] \arrow[d] & & \bullet(\bullet(\bullet\,\bullet)) \\
(\bullet(\bullet\,\bullet))\,\bullet \arrow[r] & \bullet((\bullet\,\bullet)\,\bullet) \arrow[ur] &
\end{tikzcd}$$

is only sufficient when representability is imposed!

Braided structure. $\forall\, \sigma \in B_n$, $b \in T_n$ we have

$$\mu_\sigma : P_b(X_1 \dots X_n; Y) \xrightarrow{\sim} P_b(X_{\sigma(1)} \dots X_{\sigma(n)}; Y)$$

s.t. $\mu_1 = id$ $\quad \mu_\sigma \mu_\tau = \mu_{\sigma\tau}$

μ_σ is compatible with compositions and associator.

Representability means $P_b(X_1 \dots X_n; Y) = \mathrm{Hom}_{\mathcal{C}}(\underline{X}_b; Y)$

where $\mathrm{Hom}_{\mathcal{C}}(X, Y) := P_{\to}(X, Y)$

From now onwards $\mathcal{C}$ is assumed to be $\mathbb{C}$-linear category with

$\mathrm{Hom}_{\mathcal{C}}(X, Y) := P_{\to}(X, Y)$.

(11.2) Introducing spaces.

For each $n \in \mathbb{N}$, we have a smooth variety $/ \mathbb{C}$. C_n with an (action of symmetric group, in braided case).

Concatenation : $C_n \times C_{k_1} \times \dots \times C_{k_n} \to C_{k_1 + \dots + k_n}$

($\exists$ morphism)

More generally we can assume we are given C_b $\forall\, b \in T_n$

and morphisms $C_b \times C_{b_1} \times \dots \times C_{b_n} \to C_{b(b_1 \dots b_n)}$

$$a_{b'b} : C_b \xrightarrow{\sim} C_{b'}$$

$S_n \curvearrowright C_b$ compatible with associator.

$\mathcal{C}$ = $\mathbb{C}$ linear category.

- $\forall\, b \in T_n$, $X_1 \dots X_n ; Y \in \mathcal{C}$ we have a quasi-coherent $\mathcal{O}_{C_n}$-module
  $$P_b(X_1, \dots, X_n ; Y) \quad \text{over } C_n.$$
- Composition is a morphism of sheaves
  $$P_b(X_1 \cdots X_n ; Y) \times \prod P_{b_i}(U_{i_1} \dots U_{i,k_i} ; X_i) \to \gamma^* P_{b(b_1 \dots b_n)}(\underline{U}, Y)$$
  $$\gamma : C_n \times C_{k_1} \times \dots \times C_{k_n} \longrightarrow C_{k_1 + \dots + k_n}$$
- $Id_X \in \Gamma(C_1, P_*(X, X))$
- Associator $a_{b', b}$ is a meromorphic (function) section on C_n
- $\mu_\sigma : P_b(X_1 \dots X_n ; Y) \to \sigma^* P_b(X_{\sigma(1)} \cdots X_{\sigma(n)} ; Y)$ mero. section / C_n.

Representability: $\forall\, b \in T_n$, $X_1 \dots X_n \in \mathcal{C}$ we have a family of objects $\{\underline{X}_b(s)\}_{s \in U \subset C_n}$ (dense open subset)

s.t. $\quad P_b(X_1 \dots X_n ; Y)_s = Hom_{\mathcal{C}}(\underline{X}_b(s), Y) \quad \forall\, s \in U \subset C_n$.

Special case: group G acting on category $\mathcal{C}$

$C_n = G^n$ $\quad$ [S_n acts by permuting the factors]

$$G^n \times G^{k_1} \times \dots \times G^{k_n} \longrightarrow G^{k_1 + \dots + k_n}$$
$$g_1 \cdots g_n, (g_{1,1} \cdots g_{1,k_1}) \cdots (g_{n,1} \cdots g_{n,k_n}) \longmapsto (g_1 g_{1,1} \cdots g_1 g_{1,k_1} \cdots g_n g_{n1} \cdots g_n g_{n,k_n})$$