Lecture 12

(12.0) Recall: we defined the notion of meromorphic pseudo (braided) monoidal categories.

Notations. T_n = set of (complete) bracketings on n letters = set of planar binary trees with n incoming and 1 outgoing edge.

Concatenation $T_n \times T_{k_1} \times \dots \times T_{k_n} \to T_{k_1+\dots+k_n}$

$$(b, b_1 \dots b_n) \longmapsto b(b_1, \dots, b_n)$$

Spaces: $\forall\, n \in \mathbb{N}$ we have a smooth (irr.) algebraic variety $C_n / \mathbb{C}$. together with a group action $S_n \circlearrowright C_n$. We are given morphisms

$$\gamma_{n,k_1\dots k_n} : C_n \times C_{k_1} \times \dots \times C_{k_n} \longrightarrow C_{k_1+\dots+k_n}$$

- $\mathcal{C}$ = a class of objects
- $\forall\, b \in T_n,\ X_1 \dots X_n ; Y \in \mathcal{C}$ we have a quasi-coherent $\mathcal{O}_{C_n}$-module $P_b(X_1, \dots, X_n ; Y)$
- Composition is a morphism of quasi coherent $\mathcal{O}_{C_n \times C_{k_1} \times \dots \times C_{k_n}}$-modules
$$P_b(X_1 \dots X_n ; Y) \times \prod_{i=1}^{n} P_{b_i}(U_{i,1} \dots U_{i,k_i} ; X_i) \to \gamma^* P_{b(b_1 \dots b_n)}(U_{11} \dots U_{n,k_n} ; Y)$$
- Id_X is a section of $P_{\to}(X ; X)$ over C_1.
- Associator is a meromorphic section of $\mathcal{H}om(P_b(\underline{X} ; Y), P_{b'}(\underline{X} ; Y))$, $a_{b'b}$.
- Commutativity $\forall\, \sigma \in B_n$ mapping to $\bar\sigma \in S_n$ under $B_n \twoheadrightarrow S_n$ we have a meromorphic section μ_σ of $\mathcal{H}om(P_b(X_1 \dots X_n ; Y), \bar\sigma^* P_b(X_{\sigma(1)} \dots X_{\sigma(n)} ; Y))$

Axioms:

- Composition is associative.
- Composition with Id_X is identity.
- $a_{b'b} = a_{bb'}^{-1}$ $\quad a_{b''b'}\, a_{b'b} = a_{b''b}$. a is compatible with compositions.
- $\mu_1 = id.$ $\quad \mu_\sigma \mu_\tau = \mu_{\sigma\tau}$ $\quad \mu$ is compatible with a and compositions.

Representability. $\forall\, b \in T_n$ $X_1 \dots X_n \in \mathcal{C}$ we have a family of objects

$$\{\underline{X}_b(s)\}_{s \in U \subset C_n} \quad \text{(dense open set)}$$

[Note: for representability. we assume $\mathcal{C}$ is a $\mathbb{C}$-linear category] such that $\forall\, Y \in \mathcal{C}$

$$P_b(X_1 \dots X_n ; Y)_s = \mathrm{Hom}_{\mathcal{C}}(\underline{X}_b(s), Y) \otimes \mathcal{O}_{C_n;s} \quad \forall\, s \in U \subset C_n .$$

Special case: G is a complex algebraic group acting on $\mathcal{C}$

$C_n = G^n$ $\qquad G^n \times G^{k_1} \times \dots \times G^{k_n} \to G^{k_1+\dots+k_n}$

$S_n \circlearrowright G^n$ by permuting factors $\qquad \left((g_i), (g_{i,j_i})_{\substack{j_i=1\dots k_i \\ i=1\dots n}} \right) \mapsto (g_i g_{i,j_i})$

Then we say $\mathcal{C}$ is a meromorphic G-braided category.

(12.1) Let us write the definition in a special situation which is relevant for us. Let G be a complex algebraic group.

Meromorphic faithful G tensor category (MFGTC) is a $\mathbb{C}$-linear category $\mathcal{C}$ together with a faithful functor to $\mathrm{Vect}_{\mathbb{C}}$ together with the following data

(1) $\forall\, b \in T_n$, $X_1 \dots X_n$ objects from $\mathcal{C}$ we have a <u>G-equivariant</u> see (12.3) <u>quasi-coherent sheaf</u> $\underline{X}_b$ over G^n such that over a dense open subset $i : U \hookrightarrow G^n$; $i^* \underline{X}_b$ is a vector bundle of objects from $\mathcal{C}$. For $n=1$, $X_e \simeq X$.

Here G acts on G^n by left multiplication.

(2) $\forall\, b \in T_n$, $\{ f_i : Y_i \to X_i \}_{i=1\dots n}$ morphisms from $\mathcal{C}$ we have a morphism of G-equivariant quasi-coherent sheaves over G^n

$\underline{f}_b : \underline{Y}_b \longrightarrow \underline{X}_b$ which over dense open subset consist of morphisms of $\mathcal{C}$. (Let us call these such morphisms as $\mathcal{C}$-morphism)

[Axiom: $f_i = id : X_i \to X_i \Rightarrow \underline{f}_b = id$]

(3) Associator: $\underline{X} = X_1 X_2 X_3 \qquad b_1 = (\bullet\bullet)\bullet \qquad b_2 = \bullet(\bullet\bullet)$

$a_{\underline{X}} \in \Gamma(G^3, \mathcal{H}om(\underline{X}_{b_1}, \underline{X}_{b_2})_{mer})$ meromorphic section of $\mathcal{C}$-sheaves.

i.e. over a dense open subset (not Zariski) $a_{\underline{X}}$ is a morphism in $\mathcal{C}$.

[Axiom : Pentagon axiom]

(4) Commutativity. Let $X_1, X_2 \in \mathcal{C}$, $b = (\cdot\cdot)$, $\sigma : G \times G \to G \times G$ flip

$$\cancel{a_{X_1 X}} \qquad C_{X_1, X_2} \in \Gamma\left(G \times G,\ \mathcal{H}om(\underline{X}, \sigma^* \underline{X}')_{\mathrm{mer}}\right)$$

where $\underline{X} = X_1 X_2$ $\quad \underline{X}' = X_2 X_1$

[Hexagon axiom]

(12.2) If we are given a rational function $\psi : A \to \mathrm{End}_{\mathbb{C}}(V)(u)$
A : unital assoc. algebra, then let D = set of poles of ψ. We can extend the trivial vector bundle $(\mathbb{C} \setminus D) \times V$ of A-modules to a quasi-coherent sheaf over $\mathbb{C}$ with A-action on sections $\mathcal{O}_{\mathbb{C}}(\infty D) \otimes V$
(but not as a vector bundle)

$\mathcal{O}_{\mathbb{C}}(\infty D)$ = sheaf of rational functions with finite order pole along D.

(12.3) Definition of G-equivariant structure on a sheaf.
Let X be a topological space with G action $G \times X \xrightarrow{\alpha} X$ and $\mathcal{F}$ be a sheaf over X. A G-equivariant str. on $\mathcal{F}$ is given by an iso

$$\psi : \pi_2^* \mathcal{F} \xrightarrow{\sim} \alpha^* \mathcal{F} \qquad \pi_2 : G \times X \to X \text{ projection}$$

s.t. $\psi|_{\{e\} \times G}$ = identity and let $\beta : G \times G \times X \underset{id \times \alpha}{\overset{m \times id}{\rightrightarrows}} G \times X \xrightarrow{\alpha} X$

Then
$$\begin{tikzcd} \pi_3^* \mathcal{F} \arrow[r, bend left, "(m \times id)^* \psi"] \arrow[r, bend right, "(1 \times \psi) \circ \psi"'] & \beta^* \mathcal{F} \end{tikzcd}$$
are equal.

(12.4) First consequence of the definition : we have a G-action on $\mathcal{C}$

For $X \in \mathcal{C}$, we have a vector bundle of objects $\underline{X}$ over a dense subset $U \subset G$ s.t. $X_e = X$. By G-equivariance $U = G$ and $g \cdot X := X_g$.

$X, Y \in \mathcal{C}$ we have a morphism $\underline{f} : \underline{X} \to \underline{Y}$

This gives G-action on $\mathcal{C}$.

(12.5) Meromorphic tensor structure (twist).

Let $\mathcal{C}$ and $\mathcal{D}$ be two mero. faithful tensor categories over G and G' resp. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be an additive functor. Assume we have a group hom (of $\mathbb{C}$-algebraic / analytic groups) $\rho : G \to G'$.

Axiom 1. $F(X_g) \simeq F(X)_{\rho(g)} \quad \forall \ X \in \mathcal{C}, g \in G$

Alternately for every $X \in \mathcal{C}$ we have a vector bundle $\underline{X}$ on G and $\underline{F(X)}$ on G'. We want this to be compatible with $\rho : G \to G'$.

Meromorphic twist is a meromorphic section over $G \times G$ of

$$\mathcal{J}_{X,Y} \in \mathcal{H}om\left((\rho \times \rho)^* F(X)F(Y), F(XY)\right)_{mer.}$$

which is a morphism in $\mathcal{D}$ (over an open set).

Axioms of twist : (1) Compatibility with associator.

[Left as an exercise]

(2) $\mathcal{J}_{X,Y}$ preserves braiding if it is compatible with commutativity constraint.

[write the axiom. exercise]

(12.6) Let $\mathcal{C} = \mathrm{Rep}_{fd}(\mathcal{U})$, $G = \mathbb{C}^\times$, $\otimes$ = standard tensor product.

Associator is trivial and $\forall\, n \in \mathbb{N}$, $\zeta_1 \cdots \zeta_n \in \mathbb{C}^\times$, $X_1 \dots X_n \in \mathcal{C}$

$\underline{X}_{\underline{\zeta}} := X_1(\zeta_1) \otimes \dots \otimes X_n(\zeta_n)$ forms a vector bundle over $(\mathbb{C}^\times)^n$ of $\mathcal{U}$-modules.

Commutativity $\quad c_{\zeta_1,\zeta_2} := \sigma \circ R(\zeta_1 \zeta_2^{-1}) : X_1(\zeta_1) \otimes X_2(\zeta_2) \to X_2(\zeta_2) \otimes X_1(\zeta_1)$

Note: G-equivariant structure comes from

$$\left(X_1(\zeta_1) \otimes X_2(\zeta_2)\right)(\zeta) = X_1(\zeta\zeta_1) \otimes X_2(\zeta\zeta_2).$$

Now if $\overset{D}{\otimes}$ = Drinfeld coproduct

$\underline{X}^{(D)}_{\underline{\zeta}} := X_1(\zeta_1) \overset{D}{\otimes} \dots \overset{D}{\otimes} X_n(\zeta_n)$ forms a vector bundle of $\mathcal{U}$-modules over a (Zariski) dense open subset of $(\mathbb{C}^\times)^n$. This extends to a quasi-coherent sheaf over $(\mathbb{C}^\times)^n$ (of $\mathcal{U}$-modules) (see (12.2))

(12.7) Theorem. $(\mathrm{Rep}_{fd}\,\mathcal{U}, \overset{D}{\otimes})$ has a meromorphic braided structure.

Hence it is a MF $\mathbb{C}^\times$-tensor category.

Concretely we need to construct an iso. $V(\zeta) \overset{D}{\otimes} W \xrightarrow{\sim} W \overset{D}{\otimes} V(\zeta)$ satisfying cabling axioms (= Hexagon axiom for trivial associator).

We give an explicit construction of $R_{0;V,W}(\zeta)$, $\mathrm{End}(V \otimes W)$-valued meromorphic function of ζ st

$$\sigma \circ R_{0;V,W}(\zeta) : V(\zeta) \overset{D}{\otimes} W \to W \overset{D}{\otimes} V(\zeta) \text{ is a morphism of}$$

$\mathcal{U}$-modules (generically in ζ: $V(\zeta) \overset{D}{\otimes} W \xrightarrow{\sim} W \overset{D}{\otimes} V(\zeta)$)

$\to$ $R_0(\zeta)$ arises as a solution to a q-difference equation (abelian)

$$R_0(p\zeta) = B(\zeta) R_0(\zeta) \qquad \left(p = q^{2mh^\vee},\ B(\zeta) \text{ a concretely given rational fn. of } \zeta\right)$$

(12.8) Definition of R_0 is given in the following steps:

Step 1. $A = (a_{ij})$ $\quad D = (d_i)$ as before. Consider the matrix

$$B(T) = \left(\frac{T^{d_i a_{ij}} - T^{-d_i a_{ij}}}{T - T^{-1}} \right)_{i,j \in I} = \left([d_i a_{ij}]_T \right)_{i,j \in I}$$

$B(T)^{-1} = \dfrac{1}{[\ell]_T} C(T)$ where entries of $C(T)$ are Laurent polynomials in T over $\mathbb{Z}$. $\quad (\ell = m h^\vee)$

$$c_{ij}(T) = \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} T^r \qquad \text{Note: } c_{ij}(T) = c_{ji}(T) = c_{ij}(T^{-1})$$

Step 2. Define an operator

$$\mathcal{B}(\zeta) = \exp\left[-\sum \oint_{C_1} c_{ij}^{(r)} \frac{d}{dw} \log \psi_i(w) \otimes \log \psi_j(q^{\ell+r} w \zeta)\, dw \right]$$

$\sum$ is over $i,j \in I$ and $r \in \mathbb{Z}$ (finite since $c_{ij}^{(r)} \neq 0$ for finitely many $r \in \mathbb{Z}$)

Step 3. $\quad R_0(q^{2\ell}\zeta) = \mathcal{B}(\zeta) R_0(\zeta)$

Our first goal is to make sense of $\mathcal{B}(\zeta)$. For this we need some linear algebra and matrix logarithms.

(12.9) Lemma. Let $M(u)$ be a rational ~~End(V)~~ $GL(V)$-valued function of u. (V a finite-dim'l vector space over $\mathbb{C}$)

Assume $[M(u), M(v)] = 0$

Then the semisimple and unipotent components of the Jordan dec. of M are rational functions of u

Proof. Let $\mathfrak{a} \subset \mathrm{End}\,V$ be the span of $\{M(u) : u \in \mathbb{C}\}$. Then $\mathfrak{a}$ is an abelian Lie subalgebra of $\mathrm{End}(V)$ and hence $V = \bigoplus_{\lambda \in \mathfrak{a}^*} V_\lambda$

$$V_\lambda = \{ v \in V \mid (x - \lambda(x))^N v = 0 \quad \forall x \in \mathfrak{a},\ N \gg 0 \}$$

Define
$$M_S(u) = \sum_\lambda (\text{projection onto } \lambda) \circ \lambda \circ M(u)$$
$$M_U(u) = M_S(u)^{-1} M(u)$$
$\square$

(12.10) Matrix logarithm

Note: $\psi_i^+(z) = K_i \exp\left((q_i - q_i^{-1}) \sum_{n \geq 1} H_{i,n} z^{-n} \right)$ Let $\overline{\psi}_i(z) = K_i^{-1} \psi_i(z) = \exp(H_i(z))$

$$H_i(z) = \log \overline{\psi}_i(z)$$

Prop. Let $\xi : \mathbb{C} \to \mathrm{End}(V)$ be a rational function of $u \in \mathbb{C}$ s.t. $[\xi(u), \xi(v)] = 0$ and $\xi(\infty) = 1$.

$\sigma(\xi) :=$ set of poles of $\xi(u)^{\pm 1}$

$$X(\xi) := \bigcup_{a \in \sigma(\xi)} [0,a] \qquad [0,a] = \text{line segment joining } 0 \text{ to } a.$$

Then $\log(\xi(u)) : \mathbb{C} \setminus X(\xi) \to \mathrm{End}(V)$ is a single-valued holomorphic function (commutes at different values of $u \in \mathbb{C} \setminus X(\xi)$) takes value 0 at ∞.
$$\frac{d}{du} \log \xi(u) = \xi(u)^{-1} \xi'(u)$$

Proof We only need to prove this for semi-simple case since log of a unipotent matrix can always be defined as a polynomial

$$\log(\xi_U(u)) = \sum_{k \geq 1} (-1)^{k-1} \frac{(\xi_U(u) - 1)^k}{k}$$

For s.s. case we have to make sense of $\log(1 - au^{-1})$. This is well defined for $u \notin [0,a]$. $\square$

(12.11) Now assume $A : \mathbb{C} \to \mathrm{End}(V)$ $\quad B : \mathbb{C} \to \mathrm{End}(W)$

- $[A(u), A(v)] = 0 = [B(u), B(v)]$ — irrelevant for the defn. below
- Both A and B are regular at ∞ and 0 (and invertible) $\quad \left[\begin{array}{ll} A(\infty) = \alpha & B(\infty) = \beta \\ \log(\alpha) & \log(\beta) \text{ are fixed.} \end{array} \right.$

$\underline{\alpha, \beta \text{ are semisimple}}$.

Define $$\mathcal{Y}(\zeta) = \exp\left[\oint_{C_1} \frac{d}{dw} \log A(w) \otimes \log B(\zeta w)\, dw \right]$$

C_1 encloses zeroes and poles of A

$\log B(\zeta w)$ is analytic within C_1

Prop. (1) $\mathcal{Y}(\zeta)$ is a rational function of ζ, regular at ∞ and 0

$$\mathcal{Y}(\infty) = \mathcal{Y}(0) = 1 \quad ; \quad [\mathcal{Y}(\zeta), \mathcal{Y}(\zeta')] = 0$$

(2) $$\mathcal{Y}(\zeta) = \exp \oint_{C_2} \log A(\zeta^{-1} w) \otimes \frac{d}{dw} \log B(w)\, dw$$

Cor. $\mathcal{B}(\zeta)$ defined in Step 2 of (12.8) has the following properties:

- $[\mathcal{B}(\zeta), \mathcal{B}(\zeta')] = 0$. $\mathcal{B}(\zeta)$ is a rat'l fn. of ζ, regular at 0 and ∞ and $\mathcal{B}(0) = \mathcal{B}(\infty) = 1$ (by (1) of Prop. above)
- $\mathcal{B}_{V_1 V_2}(\zeta) = (12) \circ \mathcal{B}_{V_2 V_1}(q^{-2l} \zeta^{-1}) \circ (12)$ (by (2) of Prop. above)
- $\mathcal{B}_{V_1(\alpha_1), V_2(\alpha_2)}(\zeta) = \mathcal{B}_{V_1, V_2}(\alpha_1 \alpha_2^{-1} \zeta)$ (change of variables)
- $\mathcal{B}_{V_1(\zeta_1) \overset{D}{\otimes} V_2 ; V_3}(\zeta_2) = \mathcal{B}_{V_1 V_3}(\zeta_1 \zeta_2)\, \mathcal{B}_{V_2 V_3}(\zeta_2)$ ($\log \psi$ is primitive for Drinfeld coproduct)

$$\mathcal{B}_{V_1, V_2(\zeta_2) \overset{D}{\otimes} V_3}(\zeta_1 \zeta_2) = \mathcal{B}_{V_1 V_3}(\zeta_1 \zeta_2)\, \mathcal{B}_{V_1 V_2}(\zeta_1)$$

(and hence $\mathcal{R}_0(\zeta)$ satisfies similar properties)

<u>Proof of Prop.</u> (1) Let us assume A is semisimple. This reduces to

scalar case : $A(u) = \alpha \prod \frac{u-a_i}{u-b_i}$ $\quad a_i, b_i \in \mathbb{C}^\times$, $\alpha \in \mathbb{C}^\times$

Then $y(\zeta) = \prod_i B(\zeta a_i) B(\zeta b_i)^{-1}$ clearly rat'l and takes value 1 at $\zeta = \infty, 0$ (B is regular there).

If A is unipotent : $A(u) = 1 + A_N(u)$

$\log A(u)$ is again rat'l vanishes at ∞ (since $A(\infty)$ is s.s. and unipot $\Rightarrow A(\infty) = 1$)

Write $\log A(u) = \sum_{\substack{j \\ n \in \mathbb{N}}} \frac{N_{j,n}}{(u-a_j)^{n+1}}$ $\quad a_j \in \mathbb{C}^\times$

$$y(\zeta) = \exp\left[\sum_{\substack{j \\ n \in \mathbb{N}}} -(n+1) N_{j,n} \otimes \frac{\partial_w^{n+1}}{(n+1)!} \log B(\zeta w)\Big|_{w=a_j} \right]$$

$\square$

(2) is integration by parts.

(12.12) Consider the difference equation

$\boxed{R(q^{2\ell}\zeta) = \mathcal{B}(\zeta) R(\zeta)}$. $(*)$ Let $p = q^{2\ell}$. We assume $|q| > 1$

Solution near 0 $\quad \mathcal{B}(\zeta) = 1 + \sum_{k \geq 1} \mathcal{B}_k^- \zeta^k \qquad \mathcal{R}^-(\zeta) = 1 + \sum_{k \geq 1} \mathcal{R}_k^- \zeta^k$

Then $\mathcal{R}_k^- = (p^k - 1)^{-1} \sum_{\ell=1}^{k} \mathcal{B}_\ell^- \mathcal{R}_{k-\ell}^- \qquad (\forall\, k \geq 1)$

Converges by our general result.

Solution near ∞ $\quad \mathcal{B}(\zeta) = 1 + \sum_{k \geq 1} \mathcal{B}_k^+ \zeta^{-k} \qquad \mathcal{R}^+(\zeta) = 1 + \sum_{k \geq 1} \mathcal{R}_k^+ \zeta^{-k}$

$\mathcal{R}_k^+ = (p^{-k} - 1)^{-1} \sum_{\ell=1}^{k} \mathcal{B}_\ell^+ \mathcal{R}_{k-\ell}^+ \qquad (\forall\, k \geq 1)$

Thus we have two solutions $\mathcal{R}^{\pm}_{0;V,W}(\zeta) := q^{\mp\Omega_0} R^{\pm}(\zeta)$

$\Omega_0 = \sum x_a \otimes x_a \in \mathfrak{h} \otimes \mathfrak{h} \qquad \{x_a\}$ o.n. basis of $\mathfrak{h}$.

(12.13) $R_0(\zeta)$ intertwines $\Delta_\zeta = \tau_\zeta \otimes 1\, \Delta_1$ to $\tau_\zeta \otimes 1\, \Delta_1^{op}$.

We need to establish commutation relations of $\mathcal{B}(\zeta)$ with a typical contour integral.

Set up: V_1 and V_2 are f.d. repns of $U_q(L\mathfrak{g})$

$C_{i,k}$ are contours enclosing poles of $E_k(w)$ on V_i $(i=1,2)$

$a_i : \mathbb{C} \to \mathrm{End}(V_i)$ are meromorphic functions, which are holomorphic within $C_{i,k}$ and take values in the commutative subalg. gen. by $\{\psi^{\pm}_{j,\pm n}\}$.

$$X_k^{(1)} := \oint_{C_{1,k}} a_1(v)\, E_k(v) \otimes a_2(v)\, dv$$

$$X_k^{(2)} := \oint_{C_{2,k}} a_1(v) \otimes a_2(v)\, E_k(v)\, dv$$

Prop. $$\mathrm{Ad}(\mathcal{B}_{V_1V_2}(\zeta)) \cdot X_k^{(1)} = \oint_{C_{1,k}} a_1(v)\, E_k(v) \otimes a_2(v)\, \psi_k(q^{2\ell} v \zeta)\, \psi_k(v\zeta)^{-1}\, dv$$

$$\mathrm{Ad}(\mathcal{B}_{V_1V_2}(\zeta)) \cdot X_k^{(2)} = \oint_{C_{2,k}} \psi_k(v\zeta^{-1})\, \psi_k(v\zeta^{-1} q^{-2\ell})^{-1} a_1(v) \otimes a_2(v)\, E_k(v)\, dv$$

Proof. Step 1. We have the following relation in $U_q(L\mathfrak{g})$ [see (12.15) below for a proof]

$$\mathrm{ad}\left(\frac{d}{dw_1} \log \psi_i(w_1)\right) \cdot E_k(w_2) = \left(\frac{1}{w_1 - q_i^{-a_{ik}} w_2} - \frac{1}{w_1 - q_i^{a_{ik}} w_2}\right) E_k(w_2)$$

$$- \frac{w_2}{w_1(q_{ik} w_1 - w_2)} E_k(q_{ik} w_1) + \frac{w_2\, q_{ik}}{w_1(w_1 - q_{ik} w_2)} E_k(q_{ik}^{-1} w_1)$$

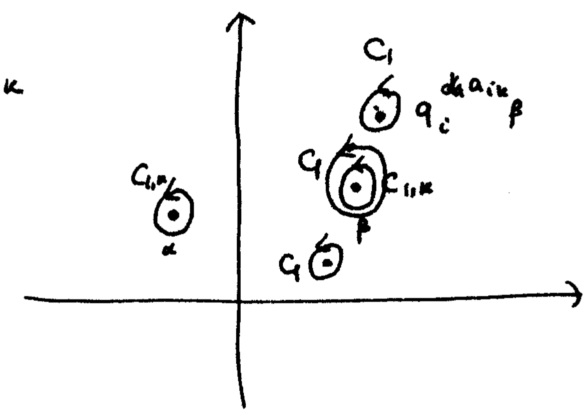
Hence if α is a pole of $E_k(w)$, $q_i^{\pm a_{ik}} \alpha$ are zeroes or poles of $\psi_i(w)$

Step 2. Using the relation above we get

$$\mathrm{ad}\ \oint_{C_1} \frac{d}{dw}\log\psi_i(w)\otimes\log\psi_j(q^{l+r}w\zeta)\,dw\ .\ X_k^{(1)}$$

$$= \oint_{C_{1,k}} a_1(v)E_k(v)\otimes a_2(v)\left(\log\psi_j\left(q^{l+r-d_ia_{ik}}v\zeta\right)-\log\psi_j\left(q^{l+r+d_ia_{ik}}v\zeta\right)\right)dv$$

Here we chose C_1 to be outside of $C_{1,k}$



This implies

$$\mathrm{ad}\left(-\sum_{\substack{i,j\in I\\ r\in\mathbb{Z}}} c_{ij}^{(r)}\oint\frac{d}{dw}\log\psi_i(w)\otimes\log\psi_j(q^{l+r}\zeta w)\,dw\right)\cdot X_k^{(1)} =$$

$$= \oint_{C_{1,k}} a_1(v)E_k(v)\otimes\sum_{\substack{i,j\in I\\ r\in\mathbb{Z}}} c_{ij}^{(r)}\left(T^{l+r+d_ia_{ik}}-T^{l+r-d_ia_{ik}}\right)\log\psi_j(\zeta v)\ dv$$

$$= \oint_{C_{1,k}} a_1(v)E_k(v)\otimes\left(\log\psi_k(q^{2l}\zeta v)-\log\psi_k(\zeta v)\right)dv$$

Since $\sum_j\sum_{\substack{i\\ r\in\mathbb{Z}}} c_{ij}^{(r)}T^r\left(T^{d_ia_{ik}}-T^{-d_ia_{ik}}\right) = \delta_{jk}\left(T^l-T^{-l}\right)$

Here $Tf(v) = f(qv)$ is the q-shift operator. $\square$

(12.14) Set $R_0^+(\zeta) = q^{-\Omega_0}\, B(\zeta)^{-1}\, B(q^{2\ell}\zeta)^{-1}\cdots$

$$R_0^-(\zeta) = q^{\Omega_0}\, B(q^{-2\ell}\zeta)\, B(q^{-4\ell}\zeta)\cdots$$

Then in the notations of section (12.13) above

$$\mathrm{Ad}\, R_0^+(\zeta)\quad X_k^{(1)} = \oint_{C_{1,k}} a_1(v)\, E_k(v) \otimes q_k^{-H_{k,0}} \underbrace{\lim_{n\to\infty} \psi_k(v\zeta)\, \psi_k(q^{2\ell n} v\zeta)}\, dv$$

$$\lim_{v\to\infty} \psi_k(v) = q_k^{H_{k,0}}$$

$$= \oint_{C_{1,k}} a_1(v)\, E_k(v) \otimes \psi_k(v\zeta)\, dv$$

Similarly $\mathrm{Ad}\, R_0^+(\zeta)\ X_k^{(2)} = \oint_{C_{1,k}} a_1(v)\, \psi_k(v\zeta^{-1})^{-1} \otimes a_2(v)\, E_k(v)\, dv$

Same for $R_0^-(\zeta)$.

$$\Delta_\zeta(E_k(z)) = E_k(z\zeta^{-1})\otimes 1 + \oint_{C_2} \frac{z}{z-w}\, \psi_k(w\zeta^{-1}) \otimes E_k(w)\, \frac{dw}{w}$$

$$= \oint_{\zeta C_1} \frac{z}{z-w}\, E_k(w\zeta^{-1})\otimes 1\, \frac{dw}{w} + \oint_{C_2} \frac{z}{z-w}\, \psi_k(w\zeta^{-1}) \otimes E_k(w)\, \frac{dw}{w}$$

$$= \oint_{C_1} \frac{z}{z-w\zeta}\, E_k(w)\otimes 1\, \frac{dw}{w} + \oint_{C_2} \frac{z}{z-w}\, \psi_k(w\zeta^{-1}) \otimes E_k(w)\, \frac{dw}{w}$$

$$\xrightarrow[\mathrm{Ad}\, R_0^+(\zeta)]{} \oint_{C_1} \frac{z}{z-w\zeta}\, E_k(w) \otimes \psi_k(w\zeta)\, \frac{dw}{w} + \oint \frac{z}{z-w}\, 1\otimes E_k(w)\, \frac{dw}{w}$$

$$= \tau_\zeta \otimes 1\quad \Delta_1^{op}(E_k(z)) \quad \text{as claimed.}$$

Thm: The $GL(V_1 \otimes V_2)$-valued functions $\mathcal{R}^{\pm}_{0;V_1,V_2}(\zeta)$ have the following properties:

(1) $$[\mathcal{R}_0(\zeta), \mathcal{R}_0(\zeta')] = 0$$

(2) $$\sigma \circ \mathcal{R}_{0;V_1,V_2}(\zeta) : V_1(\zeta) \overset{D}{\otimes} V_2 \longrightarrow V_2 \overset{D}{\otimes} V_1(\zeta)$$ is a morphism of $\mathcal{U}_q(L\mathfrak{g})$-modules

(3) $$\mathcal{R}_{0;V_1(\zeta_1)\otimes V_2, V_3}(\zeta_2) = \mathcal{R}_{0;V_1,V_3}(\zeta_1\zeta_2)\,\mathcal{R}_{0;V_2,V_3}(\zeta_2)$$

$$\mathcal{R}_{0;V_1, V_2(\zeta_2)\otimes V_3}(\zeta_1\zeta_2) = \mathcal{R}_{0;V_1,V_3}(\zeta_1\zeta_2)\,\mathcal{R}_{0;V_1,V_2}(\zeta_1)$$

(4) $$\mathcal{R}^+_0(\zeta)^{-1}_{V_1,V_2} = \sigma \circ \mathcal{R}^-_0(\zeta^{-1})_{V_2V_1} \circ \sigma$$

(5) $$\mathcal{R}_{0;V_1(\alpha),V_2(\beta)}(\zeta) = \mathcal{R}_{0;V_1,V_2}(\alpha\beta^{-1}\zeta)$$

(12.15) Proof of commutation relation

$$\left[\frac{d}{dw_1}\log\psi_i(w_1),\ E_k(w_2)\right] = \left(\frac{1}{w_1 - q_{ik}^{-1}w_2} - \frac{1}{w_1 - q_{ik}w_2}\right)E_k(w_2) - \frac{w_2 q_{ik}^{-1}}{w_1(w_1 - q_{ik}^{-1}w_2)}E_k(q_{ik}w_1)$$
$$+ \frac{w_2 q_{ik}}{w_1(w_1 - q_{ik}w_2)}E_k(q_{ik}^{-1}w_1)$$

$$\begin{cases} z = w_1 \\ w = w_2 \\ q_{ik} = q_i^{a_{ik}} \end{cases}$$

$$\text{L.H.S.} = -\sum_{\substack{n\ge 1\\ r\ge 0}} (q_{ik}^n - q_{ik}^{-n})\, z^{-n-1} w^{-r} E_{k,n+r}$$

$$= -\sum_{N\ge 0} E_{k,N}\, z^{-N-1}\left(\sum_{r=0}^{N-1}(q_{ik}^{N-r} - q_{ik}^{-N+r})\, z^r w^{-r}\right)$$

$$= \sum_{N\ge 0} E_{k,N}\, z^{-N-1}\left(q^{-N}\frac{q^N z^N w^{-N} - 1}{q z w^{-1} - 1} - q^N \frac{q^{-N} z^N w^{-N} - 1}{q^{-1} z w^{-1} - 1}\right)$$

Can be easily checked to be $=$ R.H.S. □