(13.0) Set up : Let $p \in \mathbb{C}^\times$, $|p| > 1$. Consider the following equation for an unknown function $f(z) : \mathbb{C}^\times \longrightarrow GL(V)$

$$f(pz) = B(z) f(z) \quad - \quad (*)$$

(Local theory near 0) We assume $B(z)$ is a holomorphic function near 0, taking values in $GL(V)$ or $End(V)$.

$$B(z) = \sum_{n \geq 0} B_n z^n$$

We say $(*)$ is regular near 0 if $B_0 = 1$. It is regular singular (or Fuchsian) if $B_0 \in GL(V)$.

Non-resonant if $\lambda \neq \mu$ eigenvalues of $B_0 \Rightarrow \frac{\lambda}{\mu} \notin p^{\mathbb{Z}}$.

We have similar notions near ∞. Note 0 and ∞ are only fixed points for $z \longmapsto pz$.

(13.1) Local theory near 0 (analogy with differential equations)

Let $\Upsilon(z)$ be a solution (meromorphic) of $\Upsilon(pz) = B_0 \Upsilon(z)$

<u>Theorem</u>. There exists a unique formal series

$$F(z) = 1 + \sum_{n \geq 1} F_n z^n$$

such that $F(z)\Upsilon(z)$ is a solution of $(*)$. Moreover $F(z)$ is convergent near 0.

[under the hypothesis that B_0 is non-resonant]

<u>Proof</u>. $$F(pz)\Upsilon(pz) = B(z) F(z) \Upsilon(z)$$

$$\Longleftrightarrow \quad F(pz) B_0 = B(z) F(z)$$

Comparing coefficients of z^k :

$k=0$ $\qquad B_0 = B_0$ ✓

$k \geq 1$: $\qquad p^k F_k B_0 = B_0 F_k + \sum_{\ell=1}^{k} B_\ell F_{k-\ell} \qquad (F_0 = 1)$

Let X_k be an operator on $GL(V)$ or $End(V)$ defined by

$$X_k \cdot F = p^k F B_0 - B_0 F$$

Claim $\quad X_k$ is invertible $(\forall k)$ iff B_0 is non-resonant.

Proof $\quad$ if $\{\lambda_i\}$ are eigenvalues of B_0 then $\{p^k \lambda_i - \lambda_j\}$ are eigenvalues of X_k.

Hence $\qquad F_k = X_k^{-1} \left(\sum_{\ell=1}^{k} B_\ell F_{k-\ell} \right)$.

Proof of convergence was already given in Lecture 10, section (10.4). □

Regular case: $\quad B_0 = 1$, $\Upsilon(z) = 1$ (constant function)

$$F_k = (p^k - 1)^{-1} \sum_{\ell=1}^{k} B_\ell F_{k-\ell}$$

Can be solved explicitly : for every $N \geq 1$:

$$F_N = \sum_{\substack{i_1 \dots i_\ell \geq 1 \\ i_1 + \dots + i_\ell = N}} B_{i_1} \dots B_{i_\ell} \, (p^{i_\ell} - 1)^{-1} (p^{i_{\ell-1} + i_\ell} - 1)^{-1} \dots (p^{i_1 + \dots + i_\ell} - 1)^{-1}$$

$$\Rightarrow \sum_{N \geq 0} F_N z^N = B(p^{-1} z)\, B(p^{-2} z)\, B(p^{-3} z) \dots$$

$$= \overrightarrow{\prod_{n \geq 1}} B(p^{-n} z)$$

(13.2) Global Theory: $B(z)$ is a rational function of z regular at 0 and ∞.

$$B(z) = \sum_{n\geq 0} B_n^{\pm} z^{\mp n}$$ and we assume $B_0^{\pm}$ are non-resonant.

Theorem (13.1) gives two solutions of $F(pz) = B(z)F(z)$, denoted by $F^{\pm}(z)$ which are holomorphic near ∞ and 0 resp.

We can extend them to meromorphic functions on the entire complex plane using the difference equation:

$$F^-(p^n z) = B(p^{n-1}z) \cdots B(z) F^-(z)$$

$$F^+(p^{-n} z) = B(p^{-n+1}z)^{-1} \cdots B(z)^{-1} F^+(z)$$

It remains to solve q-difference equations with constant coefficients

$$F(pz) = B_0 F(z)$$

Such equations are solved using theta function.

(13.3) General results about doubly-periodic functions.

Let $\omega_1, \omega_2 \in \mathbb{C}^\times$ s.t. $\omega_2/\omega_1 \notin \mathbb{R}$

$\Gamma = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \subset \mathbb{C}$ rank 2 lattice

An elliptic function (relative to Γ) is a meromorphic function s.t

$f(z) = f(z+\omega_1) = f(z+\omega_2)$.

Let f be an elliptic function and let C be a counterclockwise oriented rectangle with end points $\{t, t+\omega_1, t+\omega_2, t+\omega_1+\omega_2\}$ s.t. f is holomorphic on C. $(t \in \mathbb{C})$

Prop. (1) # of zeroes and poles of f within C is finite.

(2) $\sum \mathrm{Res}_\alpha f = 0$ where summation is over all poles of f enclosed by C.

(3) # of zeroes = # of poles

(4) If f has no poles within C then f is constant.

(5) Let $x \in \mathbb{C}$ and consider the equation $f(u) = x$ for u within C. Then # of solutions is independent of x.

[$0 = \oint_C \frac{f'(u)}{f(u)-x}\,du$ = # of zeroes of $f(u)-x$ − # of poles of $f(u)-x$]

(6) Let $\{\alpha_1 \dots \alpha_r\}$ be zeroes of f and $\{\beta_1 \dots \beta_r\}$ be poles of f.

Then $\sum \alpha_i - \sum \beta_i \in \Gamma$.

$$\left[\ \sum \alpha_i - \sum \beta_i = \oint \frac{u f'(u)}{f(u)}\,du = \frac{1}{2\pi i}\left[\int_t^{t+\omega_1} \left(\frac{u f'(u)}{f(u)} - \frac{(u+\omega_2) f'(u)}{f(u)}\right) du + \int_t^{t+\omega_2} -\frac{u f'(u)}{f(u)} + \frac{(u+\omega_1) f'(u)}{f(u)}\,du\right]\right.$$

$$= \frac{\omega_1}{2\pi i}\int_t^{t+\omega_2} \frac{f'(u)}{f(u)}\,du + \frac{\omega_2}{2\pi i}\int_t^{t+\omega_1} \frac{f'(u)}{f(u)}\,du \in \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

winding #

(13.4) Theta function

$\omega_1 = 1 \qquad \omega_2 = \tau$ with $\mathrm{Im}(\tau) \neq 0$.

$p = e^{2\pi i \tau}$ (let us assume $|p| > 1$ (i.e. $\mathrm{Im}(\tau) < 0$))

Define $\Theta(u;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n \rho^{-\frac{n(n-1)}{2}} z^n$ $\qquad z = e^{2\pi i u}$, $\rho = e^{2\pi i \tau}$

Prop. (1) $\Theta(u;\tau)$ is a holomorphic function of $u \in \mathbb{C}$

(2) Periodicity : $\Theta(u+1;\tau) = \Theta(u;\tau)$

$$\Theta(u+\tau;\tau) = -e^{2\pi i(u+\tau)}\,\Theta(u;\tau)$$

(3) $\Theta(u;\tau) = 0$ if and only if $u = m+n\tau$ for some $m,n\in\mathbb{Z}$.

Proof. (1)
$$\Theta(u;\tau) = 1 - z + \sum_{k\geq 2} (-1)^k \rho^{-\frac{k(k-1)}{2}} \left(z^k - z^{-k+1}\right)$$

For $u \in \mathbb{C}$ s.t. $\mathrm{Im}(u) \in \left[-\frac{A}{2\pi}, \frac{A}{2\pi}\right]$ we have $|z| \in [e^{-A}, e^{A}]$

$$\Rightarrow \quad \left|z^k - z^{-k+1}\right| < 2e^{kA}$$

Ratio test:
$$\left|\rho^{-\left(\frac{(k+1)k}{2} - \frac{k(k-1)}{2}\right)}\right| e^{A} = |\rho|^{-k} e^{A} \to 0 \text{ as } k\to\infty.$$

(2) $\Theta(u+1;\tau) = \Theta(u;\tau)$ is clear

$$\Theta(u+\tau;\tau) = \sum_{n\in\mathbb{Z}} (-1)^n \rho^{-\frac{n(n-1)}{2}+n} z^n$$

$$\left[-\frac{n(n-1)}{2} + n = -\left(\frac{n^2-3n+2}{2} - 1\right) = -\frac{(n-1)(n-2)}{2} + 1\right]$$

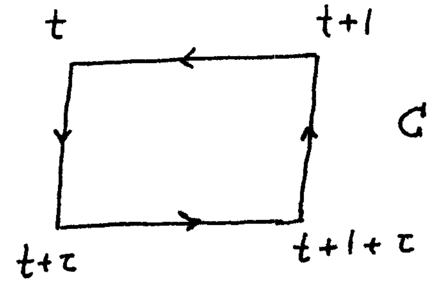
$$= \rho \sum_{n\in\mathbb{Z}} (-1)^n \rho^{-\frac{(n-1)(n-2)}{2}} z^n = -\rho z \sum_{n\in\mathbb{Z}} (-1)^{n-1} \rho^{-\frac{(n-1)(n-2)}{2}} z^{n-1}$$

$$\cancel{= -\rho z\,\Theta(u+\tau;} \quad = -\rho z\,\Theta(u;\tau)$$

(3) $\Theta(0;\tau) = 0$ clearly $\Rightarrow$ $\Theta(m+n\tau;\tau) = 0 \quad \forall\, m,n\in\mathbb{Z}$.

We claim that within a fundamental domain Θ has exactly one zero.

$$\oint_C \frac{\theta'(u)}{\theta(u)}\,du = \#\text{ zeroes} - \#\text{ poles}$$
$$= \#\text{ zeroes}$$



$$\oint_C \frac{\theta'(u)}{\theta(u)}\,du = \frac{1}{2\pi i}\int_t^{t+1} \frac{\theta'(u+\tau)}{\theta(u+\tau)} - \frac{\theta'(u)}{\theta(u)}\,du$$

$$\left[\frac{\theta'(u+\tau)}{\theta(u+\tau)} = \frac{\frac{d}{du}\left(-e^{2\pi i(\tau+u)}\theta(u)\right)}{-e^{2\pi i(\tau+u)}\theta(u)} = 2\pi i + \frac{\theta'(u)}{\theta(u)}\right]$$

$$= \frac{1}{2\pi i}\int_t^{t+1} 2\pi i\,du = 1$$

Hence Zeroes of θ $= \{m + n\tau : m, n \in \mathbb{Z}\}$ $\square$

(13.5) Solving difference equations with constant coefficients

$$F(pz) = B_0 F(z) \qquad B_0 \in GL(V)$$

Semisimple case reduces to scalar case : $f(pz) = c f(z)$ $\quad c \in \mathbb{C}^\times$

Solution is given by $\dfrac{\theta(cz)}{\theta(z)}$ $\left[\text{or } \dfrac{\theta(z)}{\theta(c^{-1}z)}\right]$

Unipotent case: let us consider the case of one Jordan block.

$$B_0 = \begin{bmatrix} 1 & \lambda & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \lambda \\ 0 & & & 1 \end{bmatrix} . \quad \lambda \in \mathbb{C}^\times .$$

That is, we choose a basis of V $\{v_1 \ldots v_N\}$ s.t.

$$B_0 v_i = v_i + \lambda v_{i-1} \qquad (i \geq 2)$$
$$B_0 v_1 = v_1$$

$$F(z)\, v_i = v_i + \sum_{k=1}^{i-1} F_{k;i}(z)\, v_{i-k}$$. Then $F(pz) = B_0\, F(z)$

becomes
$$v_i + \sum_{k=1}^{i-1} F_{k;i}(pz)\, v_{i-k} = v_i + \lambda v_{i-1} + \sum_{k=1}^{i-1} F_{k;i}(z)\left(v_{i-k} + \lambda v_{i-k-1}\right)$$

$(v_0 = 0)$

$$\left.\begin{aligned} F_{1;i}(pz) &= F_{1;i}(z) + \lambda \\ F_{2;i}(pz) &= F_{2;i}(z) + \lambda F_{1;i}(z) \\ &\vdots \\ F_{k+1;i}(pz) &= F_{k+1;i}(z) + \lambda F_{k;i}(z) \end{aligned}\right] \quad (**)$$

Soln. Let $\ell(z) = z\dfrac{\theta'(z)}{\theta(z)}$ so that $\ell(pz) = \ell(z) + 1$

For $k \geq 1$ define $$\ell_k(z) = \frac{\ell(z)\,(\ell(z)-1)\cdots(\ell(z)-k+1)}{k!} = \binom{\ell(z)}{k}$$

Then
$$\ell_{k+1}(pz) = \binom{\ell+1}{k+1} = \binom{\ell}{k+1} + \binom{\ell}{k}$$
$$= \ell_{k+1}(z) + \ell_k(z)$$

Set $F_{k;i}(z) = \lambda^k \ell_k(z)$. Then $\{F_{k;i}\}$ solve $(**)$

(13.6) Jacobi's triple product formula

Let $$F(z) = \prod_{n\geq 0}(1 - p^{-n} z) \prod_{n \geq 1} (1 - p^{-n} z^{-1})$$

Lemma. $F(z)$ is a ~~meromorphic~~ holomorphic function on $\mathbb{C}^\times$ and

$$F(pz) = -pz\, F(z)$$

Proof. Periodicity $$F(pz) = \frac{1-pz}{1-p^{-1}z^{-1}}\, F(z) = -pz\, F(z).$$

Convergence: use the following result - if $\{u_n(z)\}$ are holomorphic functions of z in a fixed domain and $|u_n(z)| < M_n$.

$$\sum_{n\geq 1} M_n \text{ converges} \Rightarrow \prod_{n \geq 1} 1 + u_n(z) \text{ converges uniformly in the domain}$$

(Proof: Choose m s.t. $\forall\, n \geq m$ $\quad M_n < \frac{1}{2}$. Then

$$\left| \frac{\log(1+M_n)}{M_n} - 1 \right| < \frac{1}{2} \Rightarrow \frac{1}{2} \leq \left| \frac{\log(1+M_n)}{M_n} \right| \leq \frac{3}{2}$$

$$\sum M_n \text{ converges} \Rightarrow \sum \log(1+M_n) \text{ converges (comparison test)})$$ $\square$

Now $\dfrac{\theta(z)}{F(z)}$ becomes a holomorphic doubly periodic function, hence a constant

$$\theta(z) = G \cdot F(z)$$

G is independent of z, only depends on $p = e^{2\pi i \tau}$.

Also as $|p| \to \infty$ $\quad \theta(z) = 1 - z \Rightarrow \lim_{|p|\to\infty} G(p) = 1$.

Thm. $\quad G(p) = \prod_{n\geq 1} 1 - p^{-n}$

Proof. Introduce $\theta_2, \theta_3, \theta_4$ as follows

$\theta_1 = \theta \qquad \theta_2(u) = \theta_1(u \pm \frac{1}{2}) \qquad \theta_3(u) = \theta_2(u - \frac{\tau}{2}) = \theta_1(u \pm \frac{1}{2} - \frac{\tau}{2})$

$\theta_4(u) = \theta_1(u - \frac{\tau}{2})$

$$\theta_1 = \sum (-1)^n e^{-n(n-1)\pi i \tau} e^{2\pi i n u} \qquad \theta_2 = \sum e^{-n(n-1)\pi i \tau} e^{2\pi i n u}$$

$$\theta_3 = \sum e^{-n^2 \pi i \tau} e^{2\pi i n u} \qquad \theta_4 = \sum (-1)^n e^{-n^2 \pi i \tau} e^{2\pi i n u}$$

θ_1 and θ_2 satisfy : $\quad \dfrac{\partial_\tau}{\pi i} = \dfrac{\partial_u}{2\pi i} - \dfrac{\partial_u^2}{(2\pi i)^2}$

θ_3 and θ_4 satisfy : $\quad \dfrac{\partial \tau}{\pi i} = - \dfrac{\partial_u^2}{(2\pi i)^2}$

Heat Equation

Triple product expressions:

$$\theta_1 = G\,(1-z) \prod_{n\geq 1} 1-p^{-n} z \prod_{n \geq 1} 1-p^{-n} z^{-1}$$

$$\theta_2 = G\,(1+z) \prod_{n\geq 1} 1+p^{-n} z \prod_{n \geq 1} 1+p^{-n} z^{-1}$$

$$\theta_3 = G \prod_{n\geq 1} 1+p^{-n+\frac{1}{2}} z \prod_{n \geq 1} 1+p^{-n+\frac{1}{2}} z^{-1}$$

$$\theta_4 = G \prod_{n\geq 1} 1-p^{-n+\frac{1}{2}} z \prod_{n \geq 1} 1-p^{-n+\frac{1}{2}} z^{-1}$$

Use the heat equation to compute logarithmic derivative wrt τ:

$\theta(0)$ means $u=0$ and hence $z=1$

$(\ )' = \frac{\partial}{\partial u}$

$$\frac{1}{\pi i} \frac{\partial_\tau \theta_1'(0)}{\theta_1'(0)} = 6 \sum_{n\geq 1} \frac{p^{-n}}{(1-p^{-n})^2}$$

$$\frac{1}{\pi i} \frac{\partial_\tau \theta_2(0)}{\theta_2(0)} = -2 \sum_{n\geq 1} \frac{p^{-n}}{(1+p^{-n})^2}$$

$$\frac{1}{\pi i} \frac{\partial_\tau \theta_3(0)}{\theta_3(0)} = -2 \sum_{n\geq 1} \frac{p^{-n+1/2}}{(1+p^{-n+1/2})^2}$$

$$\frac{1}{\pi i} \frac{\partial_\tau \theta_4(0)}{\theta_4(0)} = 2 \sum_{n\geq 1} \frac{p^{-n+1/2}}{(1-p^{-n+1/2})^2}$$

$$\Rightarrow \quad \frac{\partial_\tau \theta_1'(0)}{\theta_1(0)} = \sum_{j=2}^{4} \frac{\partial_\tau \theta_j(0)}{\theta_j(0)} \quad \Rightarrow \quad \theta_1'(0) = C\, \theta_2(0)\, \theta_3(0)\, \theta_4(0)$$

C is a constant

Let $|p| \to \infty$: $\quad -2\pi i = C\, 2 \quad \Rightarrow \quad C = -\pi i$

$$\theta_1'(0) = -\pi i\ \theta_2(0)\ \theta_3(0)\ \theta_4(0)$$

$$\Longrightarrow \quad -2\pi i\, G \prod_{n\geq 1} (1-p^{-n})^2 = -2\pi i \prod_{n\geq 1} (1+p^{-n})^2 \prod_{n \geq 1} (1-p^{-2n+1})^2\, G^3$$

$z=1$ in triple product exp.

$$\Rightarrow \quad G^2 = \prod_{n\geq 1} (1-p^{-n})^2 \quad \Rightarrow \quad G = \prod_{n \geq 1} 1-p^{-n} \qquad \square$$