(14.0) Definition. $Y_\hbar\mathfrak{g}$ is a unital associative algebra over $\mathbb{C}[\hbar]$ (or $\mathbb{C}$ if we specialize $\hbar$ to a (non-zero) complex number) generated by

$$\xi_{i,r}\ ,\ x^\pm_{i,r} \qquad (i\in I\ ,\ r\in\mathbb{N})$$

subject to the following relations.

[Y1] $[\xi_{i,r}\ \xi_{j,s}] = 0 \quad \forall\ i,j\in I,\ r,s\in\mathbb{N}$

[Y2] $[\xi_{i,0}\ ,\ x^\pm_{j,s}] = \pm d_i a_{ij}\ x^\pm_{j,s} \quad \forall\ i,j\in I\ ,\ s\in\mathbb{N}$

[Y3] $[\xi_{i,r+1}\ x^\pm_{j,s}] - [\xi_{i,r}\ x^\pm_{j,s+1}] = \pm\hbar\dfrac{d_i a_{ij}}{2}\left(\xi_{i,r}x^\pm_{j,s} + x^\pm_{j,s}\xi_{i,r}\right)$

[Y4] $[x^\pm_{i,r+1}\ x^\pm_{j,s}] - [x^\pm_{i,r}\ x^\pm_{j,s+1}] = \pm\hbar\dfrac{d_i a_{ij}}{2}\left(x^\pm_{i,r}x^\pm_{j,s} + x^\pm_{j,s}x^\pm_{i,r}\right)$

[Y5] $[x^+_{i,r}\ x^-_{j,s}] = \delta_{ij}\ \xi_{i,r+s}$

[Y6] For $i\neq j$, $m = 1-a_{ij}$. $\forall\ r_1\ldots r_m, s\in\mathbb{N}$ we have

$$\sum_{\pi\in S_m}\left[x^\pm_{i,r_{\pi(1)}}\ ,\left[x^\pm_{i,r_{\pi(2)}}\ ,\ \ldots\ \left[x^\pm_{i,r_{\pi(m)}}\ ,\ x^\pm_{j,s}\right]\cdots\right]\right] = 0$$

Remarks. (1) $Y_\hbar\mathfrak{g}$ is $\mathbb{N}$-graded by $\deg\hbar = 1$ $\deg y_r = r$ $\forall\ y = \xi_i, x^\pm_i$

(2) $$Y_\hbar\mathfrak{g}\big/\hbar Y_\hbar\mathfrak{g} \simeq \mathcal{U}(\mathfrak{g}[t]) \ :\ \begin{aligned}\overline{\xi}_{i,r} &= d_i h_i\otimes t^r\\ \overline{x}^\pm_{i,r} &= x^\pm_i\otimes t^r\end{aligned}$$

Choose $d_i^\pm\in\mathbb{C}^\times$ s.t. $d_i^+d_i^- = d_i$. Then $x_i^+ = d_i^+ e_i$ $x_i^- = d_i^- f_i$.

Generating Series:

$$\xi_i(u) = 1 + \hbar\sum_{r\geq 0}\xi_{i,r}\,u^{-r-1}$$

$$x^\pm_i(u) = \hbar\sum_{r\geq 0}x^\pm_{i,r}\,u^{-r-1}$$

[We assume $\hbar\in\mathbb{C}^\times$ from now on, so that $Y_\hbar\mathfrak{g}$ is $\mathbb{N}$-filtered]

Shift automorphism: $\forall\ s\in\mathbb{C}$ define $\tau_s\in\operatorname{Aut}Y_\hbar\mathfrak{g}$ by

$$\tau_s(y_r) = \sum_{k=0}^{r}\binom{r}{k}s^k\ y_{r-k}$$

i.e. $y(u)\longmapsto y(u-s)$ under τ_s. This is an algebra auto. by next section.

(14.1) Presentation of $Y_\hbar \mathfrak{g}$ on $\xi_i(u)$, $x_i^\pm(u)$:

- $(Y1)$ is equivalent to $[\xi_i(u), \xi_j(v)] = 0 \quad \forall\, i,j \in I$
- $(Y2)$ and $(Y3)$ are equivalent to : $\forall\, i,j \in I$, let $a = \hbar \frac{d_i a_{ij}}{2}$

$$\xi_i(u)\, x_j^\pm(v)\, \xi_i(u)^{-1} = \frac{u-v\pm a}{u-v\mp a}\, x_j^\pm(v) \mp \frac{2a}{u-v\mp a}\, x_j^\pm(u\mp a)$$

- $(Y4)$ is equivalent to :

$$x_i^\pm(u)\, x_j^\pm(v) = \frac{u-v\pm a}{u-v\mp a}\, x_j^\pm(v)\, x_i^\pm(u) + \frac{\hbar}{u-v\mp a}\left([x_{i,0}^\pm, x_j^\pm(v)] - [x_i^\pm(u), x_{j,0}^\pm] \right)$$

- $(Y5)$ is equivalent to

$$[x_i^+(u), x_j^-(v)] = \frac{\hbar}{u-v}\left(\xi_i(v) - \xi_i(u)\right)$$

- $(Y6)$ is equivalent to : $\forall\, i\neq j$, $m = 1-a_{ij}$, we have

$$\sum_{\pi\in S_m} [x_i^\pm(u_{\pi(1)}), [x_i^\pm(u_{\pi(2)}), \dots [x_i^\pm(u_{\pi(m)}), x_j^\pm(v)]\cdots] = 0$$

Proofs : $(Y1)$ $(Y5)$ and $(Y6)$ are immediate.

$(Y2)$ and $(Y3)$. Let us check the assertion for $+$ case only and hence we drop the superscript. We claim that relations $(Y2)$ and $(Y3)$ are equivalent to saying that $(u-v-a)\,\xi_i(u)\,x_j(v) - (u-v+a)\,x_j(v)\,\xi_i(u)$ is independent of v.

Coeff of u : $1-1=0$ $\qquad$ Coeff of $u v^{-s-1}$: $\hbar(x_{j,s} - x_{j,s}) = 0$

Coeff of v^{-s-1} : $\hbar^2 [\xi_{i,0}\, x_{j,s}] - 2a\hbar\, x_{j,s}$

This is 0 $\iff$ $[\xi_{i,0}, x_{j,s}] = \frac{2a}{\hbar} x_{j,s} \iff (Y2)$

Coeff of $u^{-r-1} v^{-s-1}$: $\hbar^2\left([\xi_{i,r+1}, x_{j,s}] - [\xi_{i,r}, x_{j,s+1}] - a\,(\xi_{i,r}x_{j,s} + x_{j,s}\xi_{i,r})\right)$

$= 0 \iff (Y3)$.

Set $v = u-a$ to get $(u-v-a)\,\xi_i(u)\,x_j(v) - (u-v+a)\,x_j(v)\,\xi_i(u) = -2a\, x_j(u-a)\,\xi_i(u)$.

$(Y4)$ Again let us prove it for the $+$ case.

Multiply $[x_{i,r+1}\ x_{j,s}] - [x_{i,r}\ x_{j,s+1}] = a\,(x_{i,r}x_{j,s} + x_{j,s}x_{i,r})$
by $\hbar^2 u^{-r-1} v^{-s-1}$ and sum over all $r,s \in \mathbb{N}$. Using

$$\hbar \sum_{r\geq 0} x_{r+1} u^{-r-1} = u\,(x(u) - \hbar x_0 u^{-1}) = u\,x(u) - \hbar x_0 \quad \text{we get}$$

$$[u\,x_i(u) - \hbar x_{i,0},\ x_j(v)] - [x_i(u),\ v\,x_j(v) - \hbar x_{j,0}] = a\,\big(x_i(u)x_j(v) + x_j(v)x_i(u)\big)$$

$$\Rightarrow\quad (u-v-a)\,x_i(u)x_j(v) - (u-v+a)\,x_j(v)x_i(u) = \hbar\,\big([x_{i,0}, x_j(v)] - [x_i(u), x_{j,0}]\big)$$

(14.2) Levendorskii's presentation of $Y_\hbar\mathfrak{g}$:

Generators: $\{t_{i,0},\ t_{i,1},\ x^{\pm}_{i,0},\ x^{\pm}_{i,1}\}_{i\in I}$

Relations: (L1) $[t_{i,r}\ t_{j,s}] = 0 \quad \forall\, i,j\in I,\ r,s\in\{0,1\}$.

(L2) $[t_{i,0}\ x^{\pm}_{j,s}] = \pm d_i a_{ij}\, x^{\pm}_{j,s} \quad \forall\, i,j \in I,\ s\in\{0,1\}$.

(L3) $[t_{i,1}\ x^{\pm}_{j,0}] = \pm d_i a_{ij}\, x^{\pm}_{j,1}$

(L5) $[x^+_{i,0}\ x^-_{j,0}] = \delta_{ij}\, t_{i,0} \qquad [x^+_{i,1}\ x^-_{j,0}] = [x^+_{i,0}, x^-_{j,1}] = \delta_{ij}\big(t_{i,1} + \frac{\hbar}{2} t_{i,0}^2\big)$

(L4) $[x^{\pm}_{i,1}, x^{\pm}_{j,0}] - [x^{\pm}_{i,0}\ x^{\pm}_{j,1}] = \pm\hbar \dfrac{d_i a_{ij}}{2}\,(x^{\pm}_{i,0}x^{\pm}_{j,0} + x^{\pm}_{j,0}x^{\pm}_{i,0})$

(L7) $[[t_{i,1}, x^+_{i,1}], x^-_{i,1}] + [x^+_{i,1}, [t_{i,1}, x^-_{i,1}]] = 0$

(L6) $(\mathrm{ad}\ x^{\pm}_{i,0})^{1-a_{ij}} \cdot x^{\pm}_{j,0} = 0$

Isomorphism between two presentations $\quad x^{\pm}_{i,r} \longleftrightarrow X^{\pm}_{i,r} \quad r=0,1$

$$t_{i,1} = \xi_{i,1} - \frac{\hbar}{2}\xi_{i,0}^2 \qquad t_{i,0} \longleftrightarrow \xi_{i,0}.$$

(14.3) Coproduct on $Y_\hbar\mathfrak{g}$: The following extends to a unique alg. hom

$$\Delta : Y_\hbar\mathfrak{g} \longrightarrow Y_\hbar\mathfrak{g} \otimes Y_\hbar\mathfrak{g}$$

$$\Delta(y_0) = y_0 \otimes 1 + 1 \otimes y_0 \qquad (\text{for } y = \xi_i, x_i^\pm)$$

$$\Delta(t_{i,1}) = t_{i,1} \otimes 1 + 1 \otimes t_{i,1} - \hbar \sum_{\beta>0} (\beta, \alpha_i)\, x_{\beta,0}^- \otimes x_{\beta,0}^+$$

Note : (i) $U\mathfrak{g} \hookrightarrow Y_\hbar\mathfrak{g}$ via $y_i \longmapsto y_{i;0}$. Thus we have elements $x_{\beta,0}^\pm \in Y_\hbar\mathfrak{g}$ $\forall\ \beta > 0$.

(ii) Together with $\{t_{i,1}\}_{i\in I}$, $U\mathfrak{g}$ generates $Y_\hbar\mathfrak{g}$. Hence if Δ extends to an algebra homomorphism, it is uniquely determined by expressions given above. However proof that Δ does extend is non-trivial and uses Levendorskii's presentation

(14.4) Rationality property. On a f.d. repn (π, V) of $Y_\hbar\mathfrak{g}$, the formal series $\pi(\xi_i(u))$, $\pi(x_i^\pm(u))$ $\in \operatorname{End} V [\![u^{-1}]\!]$ are Taylor series expansions of rational fns. of u (End V - valued).

Proof $\quad \operatorname{ad} t_{i,1} \cdot x_{i,r}^\pm = \pm 2 d_i\, x_{i,r+1}^\pm$

$$\Rightarrow \pm \frac{\operatorname{ad} t_{i,1}}{2d_i} \cdot x_i^\pm(u) = u\, x_i^\pm(u) - \hbar\, x_{i,0}^\pm$$

$$\Rightarrow \left(u \mp \frac{\operatorname{ad} t_{i,1}}{2d_i}\right) x_i^\pm(u) = \hbar\, x_{i,0}^\pm$$

$$\Rightarrow \quad x_i^\pm(u) = \left(u \mp \frac{\operatorname{ad} t_{i,1}}{2d_i}\right)^{-1} \cdot \hbar\, x_{i,0}^\pm .$$

If V is f.d. $\operatorname{ad} t_{i,1} \circlearrowright \operatorname{End} V$ is an operator on a f.d. space and hence $\left(u \mp \frac{\operatorname{ad} t_{i,1}}{2d_i}\right)^{-1}$ is a rat'l fn. of u. (End (End V) - valued).

Finally $\quad \xi_i(u) = 1 + [x_i^+(u), x_{i,0}^-]$ $\square$

(14.5) Proof of Levendorskii's presentation.

Let us denote the algebra defined in (14.2) by Y_L.

Easy part : $Y_L \longrightarrow Y_\hbar \mathfrak{g}$ extends to alg. hom.

$$x_{i,r}^{\pm} \longmapsto x_{i,r}^{\pm} \quad (r=0,1)$$
$$t_{i,0} \longmapsto \xi_{i,0}$$
$$t_{i,1} \longmapsto \xi_{i,1} - \frac{\hbar}{2}\xi_{i,0}^2$$

Converse $\rho : Y_\hbar\mathfrak{g} \longrightarrow Y_L$
$$\rho(x_{i,r}^{\pm}) = x_{i,r}^{\pm} \quad (r=0,1)$$
$$\rho(\xi_{i,1}) = t_{i,1} + \frac{\hbar}{2} t_{i,0}^2$$

Construct $\rho(x_{i,r}^{\pm})$ inductively by $\rho(x_{i,r+1}^{\pm}) = \dfrac{[t_{i,1}, \rho(x_{i,r}^{\pm})]}{\pm d_i}$

and define $\rho(\xi_{i,r})$ to be $[\rho(x_{i,r}^{+}), x_{i,0}^{-}]$.

(14.6) Elements $\{t_{i,r}\}_{i\in I, r\in\mathbb{N}}$ are defined by

$$t_i(u) = \hbar\sum_{r\geq 0} t_{i,r}\, u^{-r-1} = \log\Big(1+\hbar\sum_{r\geq 0}\xi_{i,r}\, u^{-r-1}\Big) = \log \xi_i(u)$$

Note $t_{i,r}$'s are polynomials in $\xi_{i,s}$'s. In fact

$$t_{i,r} = \sum_{n=1}^{r+1} \frac{(-1)^{n-1}\hbar^{n-1}}{n} \sum_{\substack{k_1\dots k_n\geq 0\\ k_1+\dots+k_n = r+1-n}} \xi_{i,k_1}\cdots\xi_{i,k_n} = \xi_{i,r} + \hbar(\dots)$$

Lemma: $\forall\, i,j\in I$, let $a = \pm d_i a_{ij}/2$. Then

$$[t_{i,r}\ x_{j,s}^{\pm}] = 2a\, x_{j,r+s}^{\pm} + 2\sum_{\substack{3\leq k\leq r+1\\ k \text{ odd}}} \frac{a^k\hbar^{k-1}}{r+1}\binom{r+1}{k} x_{j,r+s+1-k}^{\pm}$$

In particular $\exists$ elements $\tilde{t}_{ijn}$ expressed linear in $t_{i,r}$'s

$$\tilde{t}_{ijn} = \pm\, t_{i,n} + \text{linear exp. in } t_{i,0}\dots t_{i,n-1}$$

s.t. $[\tilde{t}_{ijn}\ x_{j,r}^{\pm}] = \pm \frac{d_i a_{ij}}{2} x_{j,r+n}^{\pm}$

Proof. $\quad \mathrm{Ad}\,\xi_i(u)\, x_{j,s} = \dfrac{u-\sigma+a\hbar}{u-\sigma-a\hbar}\, x_{j,s} \qquad (\sigma\cdot x_{j,s} = x_{j,s+1})$

$$\Rightarrow \quad [t_i(u), x_{j,s}] = \log\left(\frac{u-\sigma+a\hbar}{u-\sigma-a\hbar}\right) x_{j,s}$$

$$= \sum_{k\geq 1} \frac{(\sigma+a\hbar)^k - (\sigma-a\hbar)^k}{k}\, u^{-k}\, x_{j,s}$$

Comparing coeff of u^{-r-1} : $\quad \hbar\,[t_{i,r}, x_{j,s}] = \dfrac{(\sigma+a\hbar)^{r+1} - (\sigma-a\hbar)^{r+1}}{r+1}\, x_{j,s}$

and the claim follows $\square$

(14.7) We have the following relations already

$[\xi_{i,0}\ \xi_{j,\ell}] = 0$

$[\xi_{i,0}\ x^{\pm}_{j,\ell}] = \pm\, \mathrm{diag}\ x^{\pm}_{j,\ell}$

$[\xi_{i,1}\ x^{\pm}_{j,\ell}] = \pm\, \mathrm{diag}\ x^{\pm}_{j,\ell+1} \pm \mathrm{diag}\,\hbar/2\ (\xi_{i,0}\, x^{\pm}_{j,\ell} + x^{\pm}_{j,\ell}\,\xi_{i,0})$

$\xi_{i,1} = [x^+_{i,1}\ x^-_{i,0}] = [x^+_{i,0}, x^-_{i,1}] \ \underset{\mathrm{ad}\, t_{i,1}}{\leadsto}\ [x^+_{i,2}, x^-_{i,0}] = [x^+_{i,1}\ x^-_{i,1}] = [x^+_{i,0}, x^-_{i,2}] =: \xi_{i,2}$

$[x^+_{i,3}, x^-_{i,0}] = \ldots = [x^+_{i,0}\ x^-_{i,3}]\ (=: \xi_{i,3})$

Also $\quad [\xi_{i,2}, \xi_{i,1}] = 0 \quad$ by (L7) . $\Rightarrow$

Let us start by proving relations (Y1) – (Y5) for $\mathfrak{g} = \mathfrak{sl}_2$. (hence I drop the subscript i)

By (L4) for $i=j \qquad [x_1\, x_0] = \hbar\, x_0^2 \ \underset{\mathrm{ad}\, t_1}{\leadsto}\ [x_2, x_0] = \hbar\,(x_0x_1 + x_1x_0)$

(stay with + case)

$$[\xi_2, x_0^+] + [x_2^+, \xi_0] = \hbar\,(\xi_0 x_1^+ + x_0^+\xi_1 + \xi_1 x_0^+ + x_1^+\xi_0)$$

$$\underset{\mathrm{ad}\, x_0^-}{\leadsto} \qquad = [\xi_1, x_1^+] - [\xi_0, x_2^+] + \hbar\,(\xi_1 x_0^+ + x_0^+\xi_1)$$

$$\Rightarrow \quad [\xi_2, x_0^+] - [\xi_1, x_1^+] = \hbar\,(\xi_1 x_0^+ + x_0^+\xi_1)$$

$\mathrm{ad}\, t_1$ repeatedly $\Rightarrow \quad [\xi_2, x_\ell] - [\xi_1\, x_{\ell+1}] = \hbar\,(\xi_1 x_\ell + x_\ell \xi_1) \quad$ (since ξ_2, ξ_1 commute with t_1)

Lemma. $\quad [x_{k+1}\, x_\ell] - [x_k\, x_{\ell+1}] = \hbar\,(x_k x_\ell + x_\ell x_k) \quad \forall\, k, \ell \in \mathbb{N}$

Proof. We already have t_1 and $\tilde{t}_2$

$$Q(k,\ell) = [x_{k+1}\ x_\ell] - [x_k\ x_{\ell+1}] - \hbar\,(x_k x_\ell + x_\ell x_k)$$

Apply $\operatorname{ad} t_1$ twice to get

$$Q(k,\ell)=0 \Rightarrow Q(k+2,\ell)+2Q(k+1,\ell+1)+Q(k,\ell+2)=0$$

$\operatorname{ad}\tilde{t}_2$ gives $\quad Q(k,\ell)=0 \Rightarrow Q(k+2,\ell)+Q(k,\ell+2)=0$

hence $\quad Q(k,\ell)=0 \Rightarrow Q(k+1,\ell+1)=0$ & $Q(k,k)=0 \Rightarrow Q(k+2,k)=0$

We know that $Q(0,0)$, $Q(1,0)$ are 0 (and $Q(\ell,k)=+Q(k,\ell)$) □

This implies $Q(k,\ell)=0 \ \forall k,\ell\in\mathbb{N}$ by induction

Let me just give the induction step

$$[X_3^+ X_0^-] = \dots = [X_0^+, X_3^-] \underset{\operatorname{ad} t_1}{\Longrightarrow} \begin{aligned}&[X_4^+ X_0^-]-[X_3^+ X_1^-] = [X_3^+ X_1^-]-[X_2^+ X_2^-]\\ &= [X_2^+ X_2^-]-[X_1^+ X_3^-]\cdots = [X_1^+, X_3^-]-[X_0^+ X_4^-]\end{aligned}$$

$$0=[\xi_2,\tilde{t}_2] = [[X_2^+ X_0^-],\tilde{t}_2] \Rightarrow [X_4^+ X_0^-] = \dots = [X_0^+ X_4^-]$$

$$= [X_4^+ X_0^-]-[X_2^+ X_2^-]$$

$$[\xi_3,\xi_1] = [[X_3^+ X_0^-],t_1] = [X_4^+ X_0^-]-[X_3^+ X_1^-]=0$$

$$[\xi_3,\xi_0] = [[X_3^+ X_0^-],\xi_0] = 0$$

Let us check:

$$[\xi_3 X_1^+]-[\xi_2 X_2^+] = \hbar(\xi_2 X_1^+ + X_1^+\xi_2). \quad \text{LHS} = [[X_2^+ X_1^-]X_1^+]-[\xi_2 X_2^+]$$

$$= [[X_2^+ X_1^+],X_1^-] + \cancel{[X_2^+,\xi_2]} - \cancel{[\xi_2 X_2^+]}$$

$$= \hbar[X_1^+X_1^+, X_1^-] = \hbar(\xi_2 X_1^+ + X_1^+\xi_2).$$

Relations for sl_2 are proved by induction on $k+\ell=s$ in

$$[X_k^+ X_\ell^-] = [X_s^+, X_0^-] = \cdots$$

$$[\xi_k,\xi_\ell]=0$$

$$[\xi_k X_\ell] = [\xi_{k-1} X_{\ell+1}] + \hbar(\xi_{k-1}X_\ell + X_\ell\xi_{k-1}).$$

(14.8) Arbitrary $\mathfrak{g}$. Let us start by proving (Y4) $\quad a = d_i a_{ij}\hbar/2$

$$\cancel{[x_{i,k+1}]}$$

$$Q(i,j \mid k,\ell) = [x_{i,k+1}\, x_{j,\ell}] - [x_{ik}, x_{j,\ell+1}] - a\,(x_{i,k} x_{j,\ell} + x_{j,\ell} x_{i,k})$$

Apply $\operatorname{ad} t_{i,1}$ $\quad 2d_i\, Q(k+1,\ell) + d_i a_{ij}\, Q(k,\ell+1) = 0$

Apply $\operatorname{ad} t_{j,1}$ $\quad d_i a_{ij}\, Q(k+1,\ell) + 2d_j\, Q(k,\ell+1) = 0$

$\begin{bmatrix} 2d_i & d_i a_{ij} \\ d_j a_{ji} & 2d_j \end{bmatrix}$ is invertible $\Rightarrow$ $Q(k+1,\ell) = 0 = Q(k,\ell+1)$.

Similarly for $[x^+_{i,k}\ x^-_{j,\ell}] = 0$ $\quad (i \neq j) \quad$ (Y5).

Now (Y3) : $\quad [\xi_{i,k+1}\, x_{j,\ell}] = [\xi_{i,k}\, x_{j,\ell+1}] + a\,(\xi_{i,k} x_{j,\ell} + x_{j\ell}\, \xi_{ik})$

+ case

$$\text{LHS} = [[x^+_{i,k+1}, x^-_{i,0}], x^+_{j,\ell}]$$

$$= [[x^+_{i,k+1}\ x^+_{j,\ell}], x^-_{i,0}] = [[x^+_{i,k}\ x^+_{j,\ell+1}], x^-_{i,0}] + a\,([x^+_{ik} x^+_{j\ell} + x^+_{j,\ell} x^+_{i,k}, x^-_{i,0}])$$

$$= [\xi_{i,k}, x_{j,\ell+1}] + a\,(\xi_{ik}\, x_{j,\ell} + x_{j\ell}\, \xi_{ik}).$$

(Y1) $\quad [\xi_{ik}, \xi_{j\ell}] = [\tilde{t}_{i;k}, \xi_{j,\ell}] = [\tilde{t}_{i;k}, [x^+_{j,\ell}\ x^-_{j,0}]]$

$$= [x^+_{j,k+\ell}\ x^-_{j,0}] - [x^+_{j,\ell}\ x^-_{j,k}] = 0.$$

(Y6) $\quad S(k_1 \dots k_m; \ell) = \sum_{\pi \in S_m} \operatorname{ad} x_{i,k_{\pi(1)}} \cdots \operatorname{ad} x_{i,k_{\pi(m)}} \cdot x_{j,\ell}$

$S(0\dots 0; 0) = 0 \Rightarrow S(0,\dots,0;\ell) = 0 \quad \forall\, \ell \in \mathbb{N}$ $\quad$ Same argument using $\operatorname{ad} t_{i,1}$ and $\operatorname{ad} t_{j,1}$

$S(0,\dots 0;\ell) = 0 \Rightarrow S(k_1 \dots k_m;\ell) = 0 \quad \forall\, k_1 \dots k_m \in \mathbb{N}$

$\forall \ell$

$\operatorname{ad} \tilde{t}_{i,k_1}$ $\quad S(k_1, 0 \dots 0; \ell) = 0 \xrightarrow{\operatorname{ad} \tilde{t}_{i,k_2}} S(k_1, k_2, 0 \dots 0; \ell) = 0$

$\forall k_1$

$\square$