(15.0) Drinfeld's classification of irreducible finite-dim'l repns of $Y_\hbar\mathfrak{g}$.

Definition. A representation V of $Y_\hbar\mathfrak{g}$ is said to be highest weight representation (of h.w. $\{d_{i,r} \in \mathbb{C}\}_{i\in I, r\in\mathbb{N}}$) if $\exists\ v \in V$ s.t.

(i) $x^+_{i,r} v = 0 \quad \forall i \in I, r \in \mathbb{N}$ (ii) $\xi_{i,r} v = d_{i,r} v$

(iii) $V = (Y_\hbar\mathfrak{g}).v$

Given $\underline{d} = \{d_{i,r} \in \mathbb{C}\}_{i\in I, r\in\mathbb{N}}$ define $M(\underline{d})$ (Verma module) as quotient of $Y_\hbar\mathfrak{g}$ by left ideal generated by $\{\xi_{i,r} - d_{i,r},\ x^+_{i,r}\}_{i\in I, r\in\mathbb{N}}$.

$M(\underline{d})$ has a unique max'l proper submodule $(M'(\underline{d}))$. Define $L(\underline{d}) = M(\underline{d})/M'(\underline{d})$

Theorem. (1) Every irreducible f.d. repn. of $Y_\hbar\mathfrak{g}$ is a highest weight repn (and hence iso. to $L(\underline{d})$ for some collection $\underline{d}$).

(2) $L(\underline{d})$ is finite-dim'l if and only if there exist $P_i(u) \in \mathbb{C}[u]$ monic

s.t. $$1 + \hbar \sum_{r\geq 0} d_{i,r}\, u^{-r-1} = \frac{P_i(u + d_i\hbar)}{P_i(u)}$$

Proof of (1). Let V be a f.d. irr. repn. of $Y_\hbar\mathfrak{g}$. $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$ as $\mathfrak{g}$-module.

Let $\lambda \in \mathfrak{h}^*$ be max'l weight of V (i.e. $V_\mu = 0\ \forall \mu > \lambda$)

Then $x^+_{i,r} V_\lambda = 0\ (\forall i \in I, r \in \mathbb{N})$. $\{\xi_{i,r}\}$ are commuting operators on f.d. V_λ and hence $\exists$ a joint eigenvector $v \in V_\lambda$, i.e.

$\xi_{i,r} v = d_{i,r} v$ $(\forall\ i,r)$. $V' = (Y_\hbar\mathfrak{g}).v \subset V$ is ~~proper~~ submodule.

By irr. of V, $V' = V$. □

(2) is proved by reduction to sl_2-case. Let us assume the statement for sl_2 and prove it for arbitrary $\mathfrak{g}$.

(15.1) Easy part. Assume $L(\underline{d})$ is f.d. Then so is the irr. quotient (as a $Y_{d_i\hbar}(sl_2)$-module) of $Y_{d_i\hbar}(sl_2)\cdot \mathbb{1}$ (here $\mathbb{1}$ is the h.w. vector).

Notation Y_i = subalgebra generated by $\{\xi_{i,r}, x_{i,r}^{\pm}\}_{r\geq 0} \simeq Y_{d_i\hbar}(sl_2)$.

Hence $1+\hbar\sum d_{i,r}\, u^{-r-1} = \dfrac{P_i(u+d_i\hbar)}{P_i(u)}$ for some polynomial

$P_i(u)\in\mathbb{C}[u]$ (monic).

(15.2) Now assume the existence of $\{P_i(u)\}_{i\in I}$. That is,

$$1+\hbar\sum_{r\geq 0} d_{i,r}\, u^{-r-1} = \frac{P_i(u+d_i\hbar)}{P_i(u)} \qquad (\forall\ i\in I)$$

We want to show that $L = L(\underline{d})$ is finite-dimensional.

We will prove the following two assertions ($L=\bigoplus_{\mu\in\mathfrak{h}^*} L_\mu$ as $\mathfrak{g}$-mod $\lambda\in\mathfrak{h}^*$ h.w. $\lambda(h_i)=\deg P_i$)

(15.2.1) $\forall\ \mu\leq\lambda$ s.t. $L_\mu\neq 0$ and $i\in I$, $\exists\ N>0$ s.t.

$$L_{\mu-n\alpha_i}=0 \quad \forall\ n\geq N.$$

This implies f.d. of $\mathfrak{g}$-mod. L, using W-action and the fact that dominant chamber is fundamental domain for $W \curvearrowright \mathfrak{h}^*$.

(15.2.2) $\dim L_\mu<\infty \quad \forall\ \mu\leq\lambda$

Note $Y_\hbar\mathfrak{g}$ admits triangular decomposition. Thus L_μ is a span of $x^-_{i_1,r_1}\cdots x^-_{i_k,r_k}\mathbb{1}$ where $\sum_{j=1}^k \alpha_{i_j} = \lambda-\mu$.

(15.3) Lemma. $\forall\, i \in I$, $Y_i \cdot 1$ is an irreducible repn of $Y_{d_i \hbar}(sl_2)$

Proof. Let $V = Y_i \cdot 1$. Assume on the contrary that there is a proper submodule $V' \subset V$ (as Y_i-modules)

$$V = \bigoplus_{n \geq 0} (L_{\lambda - n\alpha_i} \cap V).$$

Similarly for V'. Then there exists

$0 \neq v \in V'_{\lambda - m\alpha_i}$ $(m \geq 1)$ s.t. $x^+_{i,r} v = 0$, $\xi_{i,r} v = \alpha_{i,r} v$ $(\alpha_{i,r} \in \mathbb{C})$.

Now $x^+_{j,r} v \in L_{\lambda - m\alpha_i + \alpha_j} = 0$ $(\forall\, j \neq i)$.

Fix $\{\alpha_{i,r}\}_{r \geq 0}$ and let $W' = \left\{ v' \in V' \;\middle|\; \begin{matrix} x^+_{j,r} v' = 0 \quad \forall\, j \in I,\ r \in \mathbb{N} \\ \xi_{i,r} v' = \alpha_{i,r} v' \quad \forall\, r \in \mathbb{N} \end{matrix} \right\}$

W' is non-zero space preserved by $\{\xi_{j,r}\}_{\substack{j \in I \\ r \in \mathbb{N}}}$ and hence we can find a joint eigenvector which will be h.w. vector for L, contradicting its irreducibility. □

(15.4) Lemma. For any $r > 0$, $L_{\mu - r\alpha_i}$ is spanned by vectors of the form
$\mu \in \mathfrak{h}^*$

$$X^-_1\, x^-_{i_1,k_1}\, X^-_2\, x^-_{i_2,k_2} \cdots X^-_h\, x^-_{i_h,k_h}\, X^-_{h+1}\, 1 \qquad (*)$$

$(\lambda - \mu = \alpha_{i_1} + \dots + \alpha_{i_h})$. $k_1 \dots k_h \in \mathbb{N}$ are arbitrary and

$$X^-_p = x^-_{i, l_{1,p}} \cdots x^-_{i, l_{r_p,p}} \;;\quad l_{1,p} \cdots l_{r_p,p} \in \mathbb{N}$$

$r_1, \dots, r_{h+1} \in \mathbb{N}$ and $r_1 + \dots + r_{h+1} = r$

$$r_1, \dots, r_h \leq r^\vee = \max\{-a_{ij}\}_{\substack{j \in I \\ j \neq i}}$$

Proof: use (Y6) and induction.

(15.5) Proof of (15.2.1). By previous lemma $L_{\mu-n\alpha_i}$ is spanned by vectors of the form (*) with $r_{h+1} \geq n - h r^\vee$ ($h = \text{height}(\lambda-\mu)$). Thus if $n - h r^\vee > \lambda(h_i) = \deg P_i$ we get $X^-_{h+1} 1 = 0$ by Lemma (15.3). Hence $L_{\mu-n\alpha_i} = 0$ for $n > \lambda(h_i) + h r^\vee$.

(15.6) Proof of (15.2.2) is by induction on $h = \text{height}(\lambda-\mu)$.

$h = 0$: $L_\mu = L_\lambda$ is 1-dim'l.

$h = 1$: $L_\mu = L_{\lambda-\alpha_i} \subset Y_i \cdot 1$ is f.d. by Lemma (15.3).

(for some i)

Assume $h \geq 2$. $\lambda - \mu = \alpha_{i_1} + \dots + \alpha_{i_h}$ fix an ordering $i_1 \dots i_h$ and define $V_{i_1\dots i_h}$ = span of $\{x^-_{i_1,k_1} \cdots x^-_{i_h,k_h} 1\}_{k_1\dots k_h \in \mathbb{N}}$

It is enough to show that $V_{i_1\dots i_h}$ is finite-dim'l.

$$V_{i_1\dots i_h} = \sum_{k\geq 0} x^-_{i_1 k} V_{i_2\dots i_h}$$

Now using (Y4) $x^-_{i_1 k_1} x^-_{i_2 k_2}$ can be written as a linear combination of

$x^-_{i_1,k_1-1} x^-_{i_2,k_2+1}$, $x^-_{i_2,k_2} x^-_{i_1,k_1}$, $x^-_{i_2,k_2+1} x^-_{i_1,k_1-1}$, $x^-_{i_1 k_1-1} x^-_{i_2 k_2-1}$, $x^-_{i_2 k_2-1} x^-_{i_1 k_1-1}$

Also $V_{i_2\dots i_h}$ is spanned by $x^-_{i_2 k_2} \cdots x^-_{i_h k_h} 1$ with $k_2 \dots k_h < M$

(for some fixed $M \in \mathbb{N}$)

$$\Rightarrow \quad x^-_{i_1 k} V_{i_2\dots i_h} \subset \sum_{\substack{j = i_2 \dots i_h \\ 0 \leq l \leq M+1}} x^-_{j,l} L_{\mu+\alpha_j} + x^-_{i_1 0} L_{\mu+\alpha_{i_1}}$$

and we are done by induction. $\square$