(16.0) The aim of this lecture is to prove the classification theorem for $\mathfrak{g} = sl_2$

$Y = Y_\hbar sl_2$.

Thm. Given $\{d_r \in \mathbb{C}\}_{r\geq 0}$ let $L(\underline{d})$ be the unique irr. Y-mod with highest weight $\{d_r\}_{r\geq 0}$. Then $L(\underline{d})$ is finite-dimensional if, and only if there exists a monic $P(u) \in \mathbb{C}[u]$ s.t.

$$1 + \hbar \sum_{r\geq 0} d_r u^{-r-1} = \frac{P(u+\hbar)}{P(u)}$$

(16.1) Evaluation homomorphism. For each $a \in \mathbb{C}$ we have an algebra hom

$$ev_a : Y \longrightarrow \mathcal{U} sl_2 \quad \text{defined as}$$

$ev_a(\xi_0) = h$ $\qquad ev_a(t_1) = ah - \frac{\hbar}{2}(ef + fe)$

$ev_a(x_0^\pm) = e/f$

For $\lambda \in \mathbb{C}$ and $a \in \mathbb{C}$, let $M_\lambda(a)$ (resp. $L_\lambda(a)$) be the pull-back of the Verma module M_λ (resp. irr. module L_λ) of sl_2 under ev_a.

Recall: M_λ has basis $\{m_\lambda(r)\}_{r\geq 0}$ with sl_2-action given by

$h\, m_\lambda(r) = (\lambda - 2r)\, m_\lambda(r)$ $\qquad e m_\lambda(r) = (\lambda - r + 1)\, m_\lambda(r-1)$ $\qquad f m_\lambda(r) = (r+1)\, m_\lambda(r+1)$

$L_\lambda = M_\lambda$ if $\lambda \notin \mathbb{N}$. If $\lambda \in \mathbb{N}$ $L_\lambda = $ span of $m_\lambda(r)$ $(0 \leq r \leq \lambda)$ is $(\lambda+1)$-dim'l.

Lemma. Y-action of $M_\lambda(a)$ (or $L_\lambda(a)$) is given by the following

$$\xi(u)\, m_\lambda(r) = \frac{(u-a_0)(u-a_{\lambda+1})}{(u-a_r)(u-a_{r+1})}\, m_\lambda(r)$$

$$x^+(u)\, m_\lambda(r) = \frac{\hbar(\lambda-r+1)}{u-a_r}\, m_\lambda(r-1) \quad ; \quad x^-(u)\, m_\lambda(r) = \frac{\hbar(r+1)}{u-a_{r+1}}\, m_\lambda(r+1)$$

where $a_r = a + \frac{\hbar}{2}(\lambda - 2r + 1)$.

Proof. Using the formula for ev_a, we have:

$$t_1\, m_\lambda(r) = \left(a(\lambda-2r) - \frac{\hbar}{2}\lambda - \hbar r(\lambda-r)\right) m_\lambda(r)$$

$\Rightarrow$ $\mathrm{ad}\, t_1$ on $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C} m_\lambda(r), \mathbb{C} m_\lambda(r-1))$ is given by the scalar $2a + \hbar(\lambda - 2r+1) = 2a_r$. Hence we get $x^+(u)\, m_\lambda(r) = (u-a_r)^{-1}\hbar x_0\, m_\lambda(r)$

$= \frac{\hbar}{u-a_r}(\lambda-r+1)\, m_\lambda(r)$. Similarly for $x^-(u)$. Finally, $\xi(u) = 1 + [x^+(u) x_0^-]$.

(Recall the proof of rationality prop from previous lecture:

$$x^\pm(u) = \left(u - \frac{\mathrm{ad}\, t_1}{\pm 2}\right)^{-1} \hbar\, x_0^\pm .)$$

□

Change of notation. If we set $b = a_1 = a + \frac{\hbar}{2}(\lambda-1)$ then the formulae of this lemma become

$$\xi(u)\, m_\lambda(0) = \frac{u-b+\lambda\hbar}{u-b}\, m_\lambda(0) \qquad \xi(u)\, m_\lambda(r) = \frac{(u-b-\hbar)(u-b+\lambda\hbar)}{(u-b+(r-1)\hbar)(u-b+r\hbar)}\, m_\lambda(r)$$

$$x^+(u)\, m_\lambda(r) = \frac{\hbar(\lambda-r+1)}{u-b+(r-1)\hbar}\, m_\lambda(r-1) \qquad x^-(u)\, m_\lambda(r) = \frac{\hbar(r+1)}{u-b+r\hbar}\, m_\lambda(r+1)$$

Let us denote this repn. by $V(\lambda, b) = ev^*_{b-\frac{\hbar}{2}(\lambda-1)} L_\lambda$.

$$\sigma(V(\lambda,b)) = \begin{cases} \{b, b-\hbar, \ldots, b-(\lambda-1)\hbar\} & \text{if } \lambda\in\mathbb{N} \longleftrightarrow \text{D.P.}_{\lambda-1} = \prod_{k=0} u-b+k\hbar \\ \{b - r\hbar : r\geq 0\} & \text{if } \lambda\notin\mathbb{N} \end{cases}$$

(16.2) Coproduct $\Delta(y_0) = y_0 \otimes 1 + 1 \otimes y_0$

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1 - 2\hbar\, x_0^- \otimes x_0^+$$

extends to a unique algebra hom. $Y \longrightarrow Y \otimes Y$.

Proof. Using relations of the Yangian we get

$$\Delta(x_1^+) = x_1^+ \otimes 1 + 1 \otimes x_1^+ + \hbar\, \xi_0 \otimes x_0^+$$

$$\Delta(x_1^-) = x_1^- \otimes 1 + 1 \otimes x_1^- + \hbar\, x_0^- \otimes \xi_0$$

. Let $t_2 = \xi_2 - \hbar \xi_0 \xi_1 + \frac{\hbar^2}{3}\xi_0^3$

Then $\Delta(t_2) = t_2 \otimes 1 + 1 \otimes t_2 - 2\hbar\,(x_0^- \otimes x_1^+ + x_1^- \otimes x_0^+)$

Now we can easily check Levendorskii's relations.

Lemma. $\Delta(x^+(u)) = x^+(u) \otimes 1 + \xi(u) \otimes x^+(u) + \ldots$

where $\ldots$ consists of terms of weight $(-2k) \otimes 2k+2$ $(k \geq 1)$.

(here wt of $x_r^\pm = \pm 2$ and wt. of $\xi_r = 0$)

Proof. We need to prove that $\Delta(x_r^+) = x_r^+ \otimes 1 + 1 \otimes x_r^+ + \hbar \sum_{k=0}^{r-1} \xi_k \otimes x_{r-k-1}^+ + \cdots$

This is proved by induction on r as follows

$$2\,\Delta(x_{r+1}^+) = [t_1\, x_r^+] \otimes 1 + 1 \otimes [t_1\, x_r^+] + \hbar \sum \xi_k \otimes [t_1\, x_{r-k-1}^+]$$

$$- 2\hbar\, [x_0^-, x_r^+] \otimes x_0^+ + \cdots$$

using $[t_1\, x_r^+] = 2x_{r+1}^+$ and $[x_0^-, x_r^+] = -\xi_r$ we are done. □

(16.3) Antipode and counit. $\varepsilon(y_r) = 0$ $y = \xi, x^\pm$; $r \geq 0$

defines an algebra hom. $\varepsilon : Y \to \mathbb{C}$ called the counit.

$S(y_0) = -y_0$ $(y = \xi, x^\pm)$ $S(t_1) = -t_1 - 2\hbar\, x_0^- x_0^+$

defines algebra (and coalgebra) antihomomorphism.

(16.4) Proof of Theorem. Part I.

Assume $L(\underline{d})$ is the irreducible repn. with $1 + \hbar \sum_{r \geq 0} d_r \, u^{-r-1} = \frac{P(u+\hbar)}{P(u)}$.

Write $P(u) = \prod_{i=1}^{N} u - a_i$ and consider $V = \mathbb{C}^2_{a_1} \otimes \cdots \otimes \mathbb{C}^2_{a_N}$ (where $\mathbb{C}^2_a = ev_u^*(\mathbb{C}^2)$). Then $x_r^+ |\uparrow \ldots \uparrow\rangle = 0 \;\; \forall r \geq 0$ and

$$\xi(u) \, |\uparrow \ldots \uparrow\rangle = \prod_{i=1}^{N} \frac{u - a_i + \hbar}{u - a_i} \, |\uparrow \ldots \uparrow\rangle$$

Let V' be the submodule generated by $|\uparrow \ldots \uparrow\rangle$. By universal property of $M(\underline{d})$ we have a surjective map $\begin{array}{ccc} M(\underline{d}) & \xrightarrow{\varphi} & V' \\ 1 & \longmapsto & |\uparrow \ldots \uparrow\rangle \end{array}$ and hence

$L(\underline{d}) \simeq V' / \varphi(M'(\underline{d}))$ quotient of a f-d. space, hence finite-dimensional.

(16.5) Proof of Theorem. Part II.

The other implication follows from a very general and useful result due to A. Molev.

Let L be an irreducible highest weight repn. of Y, with highest weight vector 1 s.t. $\xi(u) \, 1 = \frac{P_1(u)}{P_2(u)} \, 1$ where $\deg P_1 = \deg P_2$. Enumerate zeroes of P_1 $(\{a_i\})_{1 \leq i \leq N}$ and P_2 $(\{b_i\})_{1 \leq i \leq N}$ s.t. the following condition holds $\forall k$:

$$\left.\begin{array}{l} \text{If } \{ b_k - a_j \}_{1 \leq j \leq k} \text{ contains positive integer multiple of } \hbar \text{, then} \\ b_k - a_k \text{ is minimal among those.} \end{array}\right\} (*)$$

Thm. $$L \simeq V(\lambda_1, b_1) \otimes \cdots \otimes V(\lambda_N, b_N)$$

where $\lambda_i = \frac{b_i - a_i}{\hbar}$ $(1 \leq i \leq N)$.

This theorem completes the classification as follows

Let $L(\underline{d})$ be a f.d. irr repn. of h.w $\{d_r\}_{r\geq 0}$. By rationality prop.

$$\xi(u).\,\mathbb{1} = \left(1+\hbar \sum d_r u^{-r-1}\right)\mathbb{1} = \frac{P_1(u)}{P_2(u)}\,\mathbb{1}.$$ Hence

$$L(\underline{d}) \simeq V(\lambda_1, b_1)\otimes \cdots \otimes V(\lambda_N, b_N).$$ Since L is f.d $\lambda_i \in \mathbb{N}$ $(\forall i)$

$$\Rightarrow \quad a_i = b_i - \lambda_i \hbar \quad (\lambda_i \in \mathbb{N}) \quad \text{and hence} \quad \frac{P_1(u)}{P_2(u)} = \frac{P(u+\hbar)}{P(u)}$$

where $$P(u) = \prod_{i=1}^{N} (u-b_i)\cdots(u-b_i+(\lambda_i-1)\hbar).$$

(16.6) Proof of Theorem (16.5). Let $L = V_0(\lambda_1,b_1)\otimes\cdots\otimes V(\lambda_N,b_N)$

Claim 1. If $\eta \in L$ is s.t. $x_r^+\eta = 0$ $(\forall r\geq 0)$ $\boxed{\text{and } \xi_r\eta = \alpha_r\eta}$ not needed ~~$(\alpha_r \in \mathbb{C})$~~ then η is a scalar multiple of $m_{\lambda_1}(0)\otimes\cdots\otimes m_{\lambda_N}(0)$.

(Hence L has only one submodule K, namely the one generated by $m_{\lambda_1}(0)\otimes\cdots\otimes m_{\lambda_N}(0)$)

Proof. Write $$\eta = \sum_{p=0}^{M} \eta_p \otimes m_{\lambda_N}(p)$$ (if $\lambda_N \in \mathbb{N}$ then $M \leq \lambda_N$) $(\eta_M \neq 0)$

$$x_0^+\eta = 0 \Rightarrow \sum_{p=0}^{M} x_0^+\eta_p \otimes m_{\lambda_N}(p) + (\lambda_N - p+1)\eta_p \otimes m_{\lambda_N}(p-1) = 0 \quad - (1)$$

Coeff. of $m_{\lambda_N}(M)$ gives $x_0^+\eta_M = 0$

Coeff of $m_{\lambda_N}(M-1)$ gives $x_0^+\eta_{M-1} = -(\lambda_N - M+1)\eta_M$ - (2)
(if $M\geq 1$)

(η is an e.v. for $t_1 \Rightarrow \eta_M$ is an eigenvector for t_1 - (3)) not needed

Using $\Delta(x_r^+)$ and the same argument as (1) we get $x_r^+ \eta_M = 0$

By induction on number of tensor factors: $\eta_M = m_{\lambda_1}(0) \otimes \cdots \otimes m_{\lambda_{N-1}}(0)$

(or a scalar multiple).

Now we show that $M = 0$. If $M \geq 1$ then apply $\Delta(x^+(u))$ from Lemma 16.2 to get

$$x^+(u)\eta_{M-1} + \xi(u)\eta_M \frac{\hbar(\lambda_N - M+1)}{u - b_N + (M-1)\hbar} = 0 \quad - \quad (**)$$

$$\eta_{M-1} \in \text{Span of } \Big\{ \underbrace{m_{\lambda_1}(0) \otimes \cdots \otimes m_{\lambda_{i-1}}(0) \otimes m_{\lambda_i}(1) \otimes m_{\lambda_{i+1}}(0) \otimes \cdots \otimes m_{\lambda_{N-1}}(0)}_{\zeta_i} \Big\}_{1 \leq i \leq N-1}$$

by (2)

$$x^+(u)\zeta_i = \left(\prod_{j=1}^{i-1} \frac{u - b_j + \lambda_j \hbar}{u - b_j} \right) \frac{\hbar}{u - b_i} \eta_M \qquad \xi(u)\eta_M = \prod_{i=1}^{N-1} \frac{u - a_i}{u - b_i} \eta_M$$

Clearing denominator in $(**)$ we get

$$\prod_{i=1}^{N-1} b_N - (M-1)\hbar - a_i = 0$$

$$\Rightarrow \exists\, j \quad \text{s.t.} \quad b_N - a_j = (M-1)\hbar \in \mathbb{N}\hbar$$

by assumption $(*)$ we get $\hbar\lambda_N = b_N - a_N \leq (M-1)\hbar$

$\lambda_N \in \mathbb{N}$ contradicts $M \leq \lambda_N$.

Claim 2. $K = L$. If K is proper submodule of L then $\mathrm{Ann}(K)$ is non-zero proper submodule of L^*, which doesn't contain the h.w. vector (see below). This contradicts Claim 1 since

$$L^* \simeq V(\lambda_1, b_1 - \hbar) \otimes \cdots \otimes V(\lambda_N, b_N - \hbar)$$

as assumption $(*)$ holds.

Using the expression for the antipode S, one can easily check that:

$$L_\lambda(a)^* \simeq L_\lambda^*(a-\hbar) \qquad - \ (\dagger)$$

As sl_2-module we can identify L_λ^* with L_λ using Shapovalov form. Hence we get the following identifications as Y-modules

$$\left(M_{\lambda_1}(a_1) \otimes \cdots \otimes M_{\lambda_n}(a_n)\right)^* \simeq M_{\lambda_n}(a_n)^* \otimes \cdots \otimes M_{\lambda_1}(a_1)^*$$

$$\simeq M_{\lambda_n}^*(a_n-\hbar) \otimes \cdots \otimes M_{\lambda_1}^*(a_1-\hbar) \simeq M_{\lambda_1}(a_1-\hbar) \otimes \cdots \otimes M_{\lambda_n}(a_n-\hbar).$$

Verification of $(\dagger)$:

M_λ^* : basis $\overline{m}_\lambda(r)$ with sl_2-action $\begin{cases} e\,\overline{m}_\lambda(r) = -(\lambda-r)\,\overline{m}_\lambda(r+1) \\ f\,\overline{m}_\lambda(r) = -r\,\overline{m}_\lambda(r-1) \\ h\,\overline{m}_\lambda(r) = -(\lambda-2r)\,\overline{m}_\lambda(r) \end{cases}$

$\Rightarrow$ t_1 acting on $M_\lambda^*(b)$ can be written as $t_1\,\overline{m}_\lambda(r) = -b(\lambda-2r) - \frac{\hbar\lambda}{2} - \hbar r(\lambda-r)$

On the other hand t_1 acting on $M_\lambda(a)^*$:

$$t_1\,\overline{m}_\lambda(r) = -a(\lambda-2r) + \frac{\hbar}{2}\lambda + \hbar r(\lambda-r) - 2\hbar(\lambda-r+1)r$$

$$= (-a+\hbar)(\lambda-2r) - \frac{\hbar\lambda}{2} - \hbar r(\lambda-r).$$