(17.0) Recall: $Y_\hbar\mathfrak{g}$ is the Yangian of a simple Lie algebra $\mathfrak{g}$. We have been studying $\mathrm{Rep}\,Y$ = category of finite-dimensional representations of $Y = Y_\hbar\mathfrak{g}$.

We have proved the following properties

(1) Let (π, V) be a finite-dimensional repn. of the Yangian $Y_\hbar\mathfrak{g}$. Then $\pi(\xi_i(u))$, $\pi(x_i^\pm(u)) \in \mathrm{End}\,V[\![u^{-1}]\!]$ are Taylor series expansions of rational functions of u.

$\sigma(V) :=$ set of poles of $\{\pi(\xi_i(u)), \pi(x_i^\pm(u))\}_{i\in I} \subset \mathbb{C}$ (finite)

(2) $$\left\{\begin{matrix}\text{Iso. classes of irr. f.d.} \\ \text{repns of } Y\end{matrix}\right\} \longleftrightarrow \left\{\underset{\text{monic}}{(P_i(u) \in \mathbb{C}[u])_{i\in I}}\right\}$$

(3) $Y_\hbar\mathfrak{g}$ is a Hopf algebra

A word about (2): $V \longleftrightarrow (P_i)$ if on the h.w. vector of V, say v_0

$$\xi_i(u)v_0 = \frac{P_i(u+d_i\hbar)}{P_i(u)}v_0 \qquad (\forall\, i\in I).$$

Similarly one can show that on the lowest weight vector of V, say $\overline{v}$

$$\xi_i(u)\overline{v} = \frac{Q_i(u-d_i\hbar)}{Q_i(u)}\overline{v} \qquad (\text{for some polynomials } Q_i(u))$$

We will also need the following property of the coproduct:

$$\Delta(\xi_i(u)) = \xi_i(u)\otimes\xi_i(u) + \text{terms of the form } (\deg<0)\otimes(\deg>0)$$

(17.1) q-characters (H. Knight).

Let $V \in \mathrm{Rep}\,Y$ and let $\underline{\gamma} = \{\gamma_{i,r} \in \mathbb{C}\}_{i\in I, r\geq 0}$. Define

$$V[\underline{\gamma}] = \left\{ v \in V \;\middle|\; \begin{array}{l} \forall i \in I, r \geq 0 \;\exists\; p > 0 \text{ s.t.} \\ (\xi_{i,r} - \gamma_{i,r})^p v = 0 \end{array} \right\}$$

generalized eigenspace of generalized eigenvalue $\underline{\gamma}$.

Prop. Given $\underline{\gamma}$, there exists a f.d. repn. V s.t. $V[\underline{\gamma}] \neq 0$ if, and only if

$$1 + \hbar \sum_{r\geq 0} \gamma_{i,r} u^{-r-1} = \frac{Q_i(u+d_i\hbar)}{Q_i(u)} \frac{R_i(u)}{R_i(u+d_i\hbar)} \quad (*)$$

for some $2|I|$ tuple of polynomials (monic) $\{Q_i, R_i\}_{i\in I}$.

Proof Assume there exist $\{Q_i, R_i\}_{i\in I}$ s.t. (*) holds. Take V_1 to be irr. repn (f.d.) with Drinfeld poly. $Q_i(u)$, and V_2 to be irr. f.d. repn with lowest weight vector $\bar{v}_2$ s.t. $\xi_i(u) = \dfrac{R_i(u)}{R_i(u+d_i\hbar)}$ on $\bar{v}_2$.

Take $V = V_2 \otimes V_1$ and $v = \bar{v}_2 \otimes v_1$ ($v_1 \in V_1$ being the h.w. vector).

Then $$\xi_i(u)\left(\bar{v}_2 \otimes v_1\right) = \frac{R_i(u)}{R_i(u+d_i\hbar)} \frac{Q_i(u+d_i\hbar)}{Q_i(u)} \bar{v}_2 \otimes v_1.$$

Conversely we reduce the statement to $\mathfrak{g} = \mathfrak{sl}_2$ in a obvious way. Moreover it is enough to prove it for irr. repn of $\mathfrak{sl}_2$, since an arbitrary repn. admits a finite composition series with irr. quotients. Finally every f.d. repn (irr) is a sub-quotient of $\mathbb{C}^2_{a_1} \otimes \cdots \otimes \mathbb{C}^2_{a_N}$ and the assertion is clear for $\mathbb{C}^2_{a_1} \otimes \cdots \otimes \mathbb{C}^2_{a_N}$:

$|\underline{s}\rangle = |s_1 \cdots s_N\rangle \quad s_i \in \{\uparrow, \downarrow\}$

$$\xi(u)\,|\underline{s}\rangle = \prod_{i=1}^{N} \frac{u - a_i + \varepsilon(s_i)\hbar}{u - a_i} |\underline{s}\rangle + \cdots$$

$\varepsilon(\uparrow) = 1, \quad \varepsilon(\downarrow) = -1.$

$\square$

(17.2) Given a ~~pair~~ collection $\{Q_i(u), R_i(u)\}_{i \in I}$ of monic polynomials corresponding to a generalized eigenvalue $\underline{\gamma}$, define

$$m(\underline{\gamma}) = \prod X_{i, a_j^{(i)}} \left(\prod X_{i, b_j^{(i)}} \right)^{-1}$$

if $Q_i(u) = \prod_j u - a_j^{(i)}$. $R_i(u) = \prod_j u - b_j^{(i)}$

$m(\underline{\gamma})$ is a monomial (Laurent) in variables $\{X_{i,a}\}_{i \in I, a \in \mathbb{C}^\cdot}$

Let $\mathcal{X}$ be the ring of Laurent polynomials in $\{X_{i,a}\}_{i \in I, a \in \mathbb{C}^\cdot}$

Define q-character of a f.d. repn. V as

$$\chi_q(V) = \sum_{\underline{\gamma}} \dim V[\underline{\gamma}] \cdot m(\underline{\gamma}). \quad \in \mathcal{X}.$$

(17.3). Theorem. (1) χ_q is a ring homomorphism

$$K(\operatorname{Rep} Y) \longrightarrow \mathcal{X}.$$

($K(\operatorname{Rep} Y)$ = Free abelian gp. gen. by iso. classes of objects of Rep Y modulo relations $[V_2] = [V_1] + [V_3]$ if we have a short exact seq. $0 \to V_1 \to V_2 \to V_3 \to 0$.)

Ring structure : $[V][W] := [V \otimes W]$

(2) χ_q is injective.

Proof. It is clear that χ_q descends to the Grothendieck group.

Since $\Delta(\xi_i(u)) = \xi_i(u) \otimes \xi_i(u)$ modulo strictly triangular operator we obtain that χ_q is a ring homomorphism.

χ_q is injective. If V is irreducible with Drinfeld polynomials $\{P_i(u)\}$

$$P_i(u) = \prod_j u - a_j^{(i)} \quad , \text{ then } \quad \chi_q(V) = \underbrace{\prod_{\substack{i \in I \\ j}} X_{i, a_j^{(i)}}}_{m(P_i)} + \cdots$$

Since $m(P_i)$ are linearly independent and $K(\mathrm{Rep}\,Y) =$ free abelian gp. on $\{[V] : V \text{ irr repn}\}$, we obtain injectivity of χ_q. $\square$

(17.4) Examples. Recall for $\mathfrak{g} = \mathfrak{sl}_2$, $\lambda \in \mathbb{N}$ $b \in \mathbb{C}$, we have a f.d. irr repn $V(\lambda, b)$ given explicitly as follows:

Basis $\{m_\lambda(r) : 0 \le r \le \lambda\}$

$Y_\hbar \mathfrak{sl}_2$ - action $\quad x^+(u)\, m_\lambda(r) = \dfrac{\hbar(\lambda - r + 1)}{u - b + (r-1)\hbar}\, m_\lambda(r-1) \qquad x^-(u)\, m_\lambda(r) = \dfrac{\hbar(r+1)}{u - b + r\hbar}\, m_\lambda(r+1)$

$$\xi(u)\, m_\lambda(r) = \frac{(u - b - \hbar)\,(u - b + \lambda\hbar)}{(u - b + (r-1)\hbar)\,(u - b + r\hbar)}\, m_\lambda(r)$$

$$\chi_q(V(\lambda, b)) = \sum_{r=0}^{\lambda} X_{b - r\hbar} \cdots X_{b - (\lambda-1)\hbar}\; X_{b+\hbar}^{-1}\, X_b^{-1} \cdots X_{b-(r-2)\hbar}^{-1}$$

Using this and Molev's result, one can compute χ_q for arbitrary irreducible f.d. repn of $Y_\hbar \mathfrak{sl}_2$.

For arbitrary $\mathfrak{g}$ this problem is open.

(17.5) Rep Y is not semisimple nor braided : example

$\mathfrak{g} = sl_2$. $V = \mathbb{C}^2_{a+\hbar} \otimes \mathbb{C}^2_a$

(1) $Y_\hbar sl_2$ action on $\mathbb{C}^2_{b_1} \otimes \mathbb{C}^2_{b_2}$:

$$|\downarrow\downarrow\rangle \xrightarrow{x^+(u) = \begin{bmatrix} \hbar/u-b_1 \\ \frac{\hbar(u-b_1-\hbar)}{(u-b_1)(u-b_2)} \end{bmatrix}} \begin{matrix} |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \end{matrix} \xrightarrow{x^+(u) = \left(\frac{\hbar(u-b_1+\hbar)}{(u-b_1)(u-b_2)} \quad \frac{\hbar}{u-b_1} \right)} |\uparrow\uparrow\rangle$$

$$\xi(u) = \frac{(u-b_1-\hbar)(u-b_2-\hbar)}{(u-b_1)(u-b_2)}$$

$$\xi(u) = \begin{bmatrix} \frac{(u-b_1+\hbar)(u-b_2-\hbar)}{(u-b_1)(u-b_2)} & 0 \\ \frac{-2\hbar^2}{(u-b_1)(u-b_2)} & \frac{(u-b_1-\hbar)(u-b_2+\hbar)}{(u-b_1)(u-b_2)} \end{bmatrix}$$

$$\xi(u) = \frac{(u-b_1+\hbar)(u-b_2+\hbar)}{(u-b_1)(u-b_2)}$$

Similarly $x^-(u)\,|\uparrow\uparrow\rangle = \frac{\hbar}{u-b_2}|\uparrow\downarrow\rangle + \frac{\hbar(u-b_2+\hbar)}{(u-b_1)(u-b_2)}|\downarrow\uparrow\rangle$

$x^-(u)\,|\uparrow\downarrow\rangle = \frac{\hbar(u-b_2-\hbar)}{(u-b_1)(u-b_2)}|\downarrow\downarrow\rangle \qquad x^-(u)\,|\downarrow\uparrow\rangle = \frac{\hbar}{u-b_2}|\downarrow\downarrow\rangle$

Special cases :

$$0 \to \mathbb{1} \longrightarrow \mathbb{C}^2_{a+\hbar} \otimes \mathbb{C}^2_a \longrightarrow L_2(a+\hbar/2) \to 0 \qquad \mathrm{I}$$

Trivial subrepn is generated by $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

$$0 \to L_2(a+\hbar/2) \to \mathbb{C}^2_a \otimes \mathbb{C}^2_{a+\hbar} \longrightarrow \mathbb{1} \to 0 \qquad \mathrm{II}$$

Three dim'l subrepn. gen. by $\{|\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle\}$

I and II are non-split short exact sequences. In particular

$$\mathbb{C}^2_{a+\hbar} \otimes \mathbb{C}^2_a \not\cong \mathbb{C}^2_a \otimes \mathbb{C}^2_{a+\hbar}$$

(17.6) R-matrix.

Theorem (Drinfeld) $\exists !$ $R(u) \in Y \otimes Y [\![u^{-1}]\!]$ of the form

$$R(u) = 1 + \hbar \Omega u^{-1} + O(u^{-2}) \quad \text{s.t.}$$

(i) $$\tau_u \otimes 1 \circ \Delta^{op}(x) = R(u) \left(\tau_u \otimes 1 \circ \Delta(x) \right) R(u)^{-1} \quad \forall\, x \in Y.$$

(ii) $$\Delta \otimes 1 \; R(u) = R_{13}(u) R_{23}(u)$$
$$1 \otimes \Delta \; R(u) = R_{13}(u) R_{12}(u)$$

(iii) $$R(u)^{-1} = R_{21}(-u)$$

(iv) $$\tau_a \otimes \tau_b \; R(u) = R(u+a-b)$$

Remarks: (1) The proof is cohomological in nature. (unpublished)

(2) The convergence of $R(u)$ on a tensor product of f.d. repns. is still not known. An argument similar to one used for quantum loop algebras will not work since crossing symmetry is an additive difference equation and formal solutions of such equations not necessarily converge.

(17.7) Drinfeld coproduct. We define a new action of $Y_\hbar \mathfrak{g}$ on $V_1 \otimes V_2$ ($V_1, V_2 \in \operatorname{Rep} Y$) assuming $\sigma(V_1) \cap \sigma(V_2) = \phi$, by

$$\xi_i(u) \longmapsto \xi_i(u) \otimes \xi_i(u)$$

$$x_i^+(u) \longmapsto x_i^+(u) \otimes 1 + \oint_{C_2} \frac{1}{u-v} \, \xi_i(v) \otimes x_i^+(v) \, dv$$

$$x_i^-(u) \longmapsto 1 \otimes x_i^-(u) + \oint_{C_1} \frac{1}{u-v} \, x_i^-(v) \otimes \xi_i(v) \, dv$$

C_j encloses $\sigma(V_j)$ and avoids $\sigma(V_{3-j})$ $(j = 1, 2)$.

The resulting repn. is denoted by $V_1 \underset{D}{\otimes} V_2$. In general we get a rational family of repns. $V_1(s) \otimes_D V_2$ with poles at $s = a_2 - a_1$ $(a_j \in \sigma(V_j))$.

Thm. $(\mathrm{Rep}\, Y, \otimes_D)$ is meromorphic braided tensor category.

Proof. Braiding is constructed by employing a method similar to one used for $\mathcal{U}_q(L\mathfrak{g})$ in Lecture 13. Namely, define

$$\mathcal{A}_{V_1,V_2}(s) := \exp\left[- \sum_{\substack{i,j \in I \\ r \in \mathbb{Z}}} c_{ij}^{(r)} \oint_{C_1} t_i'(v) \otimes t_j\left(v + \tfrac{(\ell+r)\hbar}{2}\right) dv \right]$$

where • $\ell = m h^\vee$ ($m = 1, 2$ or 3 for ADE, BCF, G types resp. and $h^\vee = 1 + \rho(\theta^\vee)$: θ is the longest root and $\rho \in \mathfrak{h}^*$, $\rho(h_i) = 1$ $\forall i \in I$)

• let $B(q) = \left([d_i a_{ij}]_q \right)_{i,j \in I}$. Then (Khoroshkin-Tolstoy)

$$B(q)^{-1} = \frac{1}{[\ell]} C(q) .$$ Entries of $C(q)$ are Laurent poly. in q

$$c_{ij}(q) = \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} q^r .$$ [One can prove that $\mathcal{A}(s) = 1 + O(s^{-2})$]

Consider the difference equation $R_0(s + 2\ell\hbar) = \mathcal{A}(s) R_0(s)$. General theory implies that $\exists!\, R_0^{0,\pm}(s)$ solutions of this

$$R^{0,+}(s) = \prod_{n \geq 0} \mathcal{A}(s + 2n\ell\hbar)^{-1} \qquad R^{0,-}(s) = \prod_{n \geq 1} \mathcal{A}(s - 2n\ell\hbar)$$

Both $R^{0,\pm}$ give braiding on $\mathrm{Rep}\, Y$, equipped with Drinfeld coproduct. That is, we have the following

(17.8) Thm (G.– Toledano Laredo).

(1) $\sigma \circ R^{0,\pm}_{V_1,V_2}(s) : V_1(s) \otimes_D V_2 \longrightarrow V_2 \otimes_D V_1(s)$ is a morphism (mero. in s).

(2) Cabling identities:

$$R^0_{V_1(s_1)\otimes V_2,\, V_3}(s_2) = R^0_{V_1 V_3}(s_1+s_2)\, R^0_{V_2 V_3}(s_2)$$

$$R^0_{V_1,\, V_2(s_2)\otimes V_3}(s_1+s_2) = R^0_{V_1 V_3}(s_1+s_2)\quad R^0_{V_1 V_2}(s_1)$$

(3)
$$R^{0,+}_{V_1 V_2}(s)^{-1} = \sigma \circ R^{0,-}_{V_2 V_1}(-s) \circ \sigma$$

Conjecture: There exists a (rational) intertwiner, natural in V_1, V_2

$$\bar{R}_{V_1 V_2}(s) : V_1(s) \otimes V_2 \longrightarrow V_1(s) \otimes_D V_2$$

satisfying the twist equation:

$$\bar{R}_{V_1(s_1)\otimes V_2,\, V_3}(s_2) \circ \bar{R}_{V_1 V_2}(s_1) \otimes \mathrm{id}_{V_3} = \bar{R}_{V_1,\, V_2(s_2)\otimes V_3}(s_1+s_2) \circ \mathrm{id}\otimes \bar{R}_{V_2 V_3}(s_2)$$

Explicit expression for $\bar{R}(s)$, for sl_2, exist (G.-)

This conjecture will imply that Drinfeld's universal R-matrix is a mero fn once evaluated on a $\otimes$ of f.d. repns.