Lecture 18

(18.0) Recall: $Y_\hbar\mathfrak{g}$ = Yangian of a simple f.d. Lie algebra $\mathfrak{g}$. Here $\hbar \in \mathbb{C}^\times$ is a deformation parameter.

$\mathrm{Rep}\,Y$ = category of f.d. repns. of $Y = Y_\hbar\mathfrak{g}$.

Last time we introduced Drinfeld coproduct which endows $\mathrm{Rep}\,Y$ with a mero. tensor structure. For $V_1, V_2 \in \mathrm{Rep}\,Y$ s.t. $\sigma(V_1)\cap\sigma(V_2)=\phi$, the Yangian acts on $V_1\otimes V_2$ via

$$\left.\begin{aligned} \xi_i(u) &\longmapsto \xi_i(u)\otimes\xi_i(u) \\ x_i^+(u) &\longmapsto x_i^+(u)\otimes 1 + \oint_{C_2}\frac{1}{u-v}\,\xi_i(v)\otimes x_i^+(v)\,dv \\ x_i^-(u) &\longmapsto 1\otimes x_i^-(u) + \oint_{C_1}\frac{1}{u-v}\,x_i^-(v)\otimes\xi_i(v)\,dv \end{aligned}\right]$$

u is outside of C_j $(j=1,2)$

C_j encloses $\sigma(V_j)$ and avoids $\sigma(V_{3-j})$

Let $V_1 \underset{D}{\otimes} V_2$ denote the resulting representation. For arbitrary V_1 and V_2 we obtain a rational family of repns. $\{V_1(s)\underset{D}{\otimes}V_2\}$ with poles at $s\in\sigma(V_2)-\sigma(V_1) := \{a_2-a_1 \mid a_j\in\sigma(V_j),\ j=1,2\}$.

(18.1) Thm. There exist meromorphic functions

$$R^{0,\pm}_{V_1,V_2}(s) : \mathbb{C} \to \mathrm{End}(V_1\otimes V_2)$$

s.t.

(1) $R^{0,\pm}_{V_1,V_2}(s)$ is holomorphic and invertible for $\pm\,\mathrm{Re}(s/\hbar)\gg 0$, and

$R^{0,\pm}(s) \sim 1 + \hbar\Omega^0 s^{-1} + \dots$ as $s\to\infty$ in $\pm\,\mathrm{Re}(s/\hbar)\gg 0$

$$[R^0(s), R^0(s')] = 0 .$$

(2) $$\sigma\circ R^{0,\pm}_{V_1,V_2}(s) : V_1(s)\underset{D}{\otimes}V_2 \longrightarrow V_2\underset{D}{\otimes}V_1(s)$$ is a morphism in $\mathrm{Rep}\,Y$.

(3) $$R^{0,+}_{V_1,V_2}(s)^{-1} = \sigma\circ R^{0,-}_{V_2V_1}(-s)\circ\sigma$$

(4) For $V_1, V_2, V_3 \in \mathrm{Rep}\,Y$

$$R^{\circ}_{V_1(s_1)\otimes V_2,\, V_3}(s_2) = R^{\circ}_{V_1V_3}(s_1+s_2)\; R^{\circ}_{V_2V_3}(s_2)$$

$$R^{\circ}_{V_1,\, V_2(s_2)\otimes V_3}(s_1+s_2) = R^{\circ}_{V_1V_3}(s_1+s_2)\; R^{\circ}_{V_1V_2}(s_1)$$

(5) $$R^{\circ}_{V_1(a),\, V_2(b)}(s) = R^{\circ}_{V_1V_2}(s+a-b)$$

Proof. We construct $R^{\circ}_{V_1V_2}(s)$ in the same way as for $U_q(L\mathfrak{g})$ case (see Lecture 12).

Step 1. Define $B(T) = \left([d_i a_{ij}]_T\right)_{i,j\in I}$. Then (Khoroshkin-Tolstoy)

$$B(T)^{-1} = \frac{1}{[\ell]_T}\, C(T) \quad \text{where} \quad \ell = m h^{\vee} : \; h^{\vee} = \text{dual Coxeter \#} = 1+\rho(\theta^{\vee})$$

($\rho \in \mathfrak{h}^*$ is s.t. $\rho(h_i)=1\ \forall i$, $\theta \in \mathfrak{h}^*$ is the longest root)

and $m = 1, 2, 3$ if $\mathfrak{g}$ is of type ADE, BCF or G resp.

$C(T)$ has entries from $\mathbb{Z}[T, T^{-1}]$, $(c_{ij}(T))_{i,j\in I}$

$$c_{ij}(T) = \sum_{r\in\mathbb{Z}} c_{ij}^{(r)}\, T^r$$

(finite sum)

Step 2. Define

$$\mathcal{A}(s) = \exp\left[- \sum_{\substack{i,j\in I\\ r\in\mathbb{Z}}} c_{ij}^{(r)} \oint_{C_1} t_i'(v) \otimes t_j\!\left(v+s+\frac{(\ell+r)\hbar}{2}\right) dv \right]$$

$$t_i(u) = \log \xi_i(u)$$

We explained in Lecture 12 how to make sense of this logarithm.

Check (see §12.11) $\mathcal{A}(s)$ is a rat'l fn. of s, regular at ∞

$$\mathcal{A}(s) = 1 - \ell\hbar^2\,\Omega_0\, s^{-2} + O(s^{-3})$$

Step 3. $R^0(s+2\ell\hbar) = \mathcal{A}(s)\, R^0(s)$

We will study additive difference equations in detail (next lecture). The one above can be solved as

$$R^{0,+}(s) = \mathcal{A}(s)^{-1}\, \mathcal{A}(s+2\ell\hbar)^{-1}\, \mathcal{A}(s+4\ell\hbar)^{-1} \cdots$$

$$R^{0,-}(s) = \mathcal{A}(s-2\ell\hbar)\, \mathcal{A}(s-4\ell\hbar)\, \mathcal{A}(s-6\ell\hbar) \cdots$$

The theorem (except for (ii)) is proved exactly as its $\mathcal{U}_q(L\mathfrak{g})$ counterpart.

(18.2) Conjecture. There exist meromorphic twist (tensor structure) on the identity functor $(\mathrm{Rep}\,Y, \otimes) \longrightarrow (\mathrm{Rep}\,Y, \underset{D}{\otimes})$, which is in fact rational.

In more detail, we have iso. of repns, rat'l in s, natural in V_1, V_2:

$$R^-_{V_1,V_2}(s) : V_1(s) \otimes V_2 \longrightarrow V_1(s) \underset{D}{\otimes} V_2$$

s.t. $\forall\ V_1, V_2, V_3 \in \mathrm{Rep}\,Y$ the following diagram is commutative

$$\begin{array}{ccc}
V_1(s_1+s_2) \otimes V_2(s_2) \otimes V_3 & \xrightarrow{\ \mathrm{Id} \otimes R^-_{V_2V_3}(s_2)\ } & V_1(s_1+s_2) \otimes V_2(s_2) \underset{D}{\otimes} V_3 \\
\| & & \downarrow R^-_{V_1,\, V_2(s_2) \underset{D}{\otimes} V_3}(s_1+s_2) \\
(V_1(s_1) \otimes V_2)(s_2) \otimes V_3 & & \\
\downarrow R^-_{V_1V_2}(s_1) \otimes \mathrm{Id} & & \\
(V_1(s_1) \underset{D}{\otimes} V_2)(s_2) \otimes V_3 & \xrightarrow{\ R^-_{V_1(s_1) \underset{D}{\otimes} V_2,\, V_3}(s_2)\ } & V_1(s_1+s_2) \underset{D}{\otimes} V_2(s_2) \underset{D}{\otimes} V_3
\end{array}$$

Conjecture is true for $\mathfrak{g} = \mathfrak{sl}_2$.

(18.3) Sliced subcategories of Rep Y.

Fix a subset $\Pi \subset \mathbb{C}$ s.t. $\Pi \pm d_i a_{ij} \frac{\hbar}{2} \subset \Pi \quad (\forall i, j \in I)$.

Definition $\mathrm{Rep}^{\Pi} Y$ is the full subcategory of Rep Y consisting of representations, the Drinfeld polynomials of irreducible factors of whose composition series have roots in Π. That is, let $V \in \mathrm{Rep}\,Y$ and let $0 = V_{-1} \subset V_0 \subset \dots \subset V_r = V$ be its composition series.

Let $\{P_i^{(j)}(u)\}_{i \in I}$ be Drinfeld poly. of V_j / V_{j-1} $(0 \leq j \leq r)$. Then

$$V \in \mathrm{Rep}^{\Pi} Y \iff \text{Zeroes of } P_i^{(j)} \subset \Pi \quad \forall \begin{smallmatrix} 0 \leq j \leq r \\ i \in I \end{smallmatrix}.$$

Theorem. Let $V \in \mathrm{Rep}\,Y$. Then the following are equivalent.

(1) $V \in \mathrm{Rep}^{\Pi} Y$.

(2) $\sigma(V) \subset \Pi$.

(3) Poles of $\xi_i(u)$ are contained in Π $(\forall i \in I)$.

(4) Poles of eigenvalues of $\xi_i(u)$ are contained in Π $(\forall i \in I)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) are clear.

We prove (1) $\Rightarrow$ (2) by induction on the length of composition series of V.

Remark. By Knight's result zeroes of $\xi_i(u)$ are obtained by shifting the poles of its diagonal entries by $\pm d_i \hbar$ and Π is stable under these shifts. So (3) (or (4)) $\Rightarrow$ $\xi_i(u)$ are regular and invertible on $\mathbb{C} \setminus \Pi$.

(18.4) **Base Case:** V is irreducible. Let $\{P_i\}_{i\in I}$ be Drinfeld polynomials of V. Since $V \in \mathrm{Rep}^{\Pi} Y$, zeroes of P_i lie in Π $(\forall i \in I)$.

To prove: $\{\xi_i(u), x_i^{\pm}(u)\}$ have poles in Π.

Let us write $V = \bigoplus_{\mu\in\mathfrak{h}^*} V[\mu]$ as a $\mathfrak{g}$-module. Let $\lambda \in \mathfrak{h}^*$ be the highest weight. We will use the following properties of V:

(P1) For a weight $\mu < \lambda$, the weight space $V[\mu]$ is spanned by $\{x_{i,r}^- V[\mu+\alpha_i]\}_{i\in I, r\in\mathbb{N}}$.

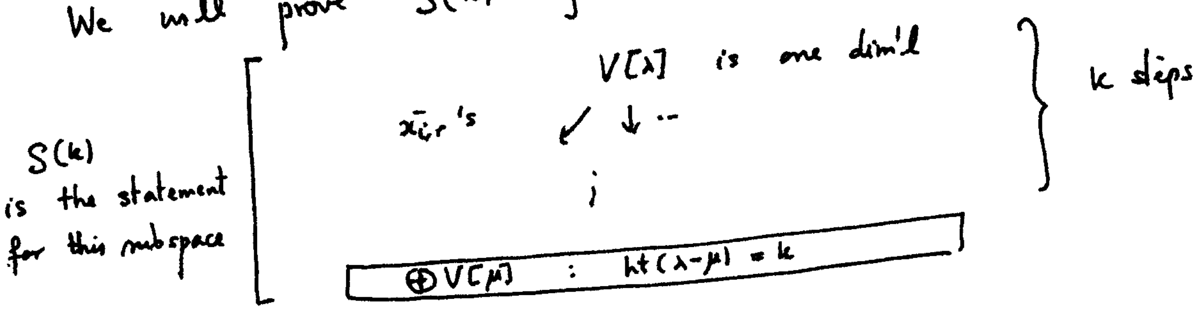
(P2) If $v \in V[\mu]$ is annihilated by $x_{i,r}^+$ $(\forall i \in I, r \in \mathbb{N})$ and $\mu < \lambda$ then $v = 0$.

Recall: $\mu < \lambda$ means $\lambda - \mu = \sum_{i\in I} n_i \alpha_i$ $(n_i \in \mathbb{N})$. Define

$$ht(\lambda-\mu) = \sum n_i .$$

$S(k)$: $\forall i \in I$, $\xi_i(u)_\mu$, $x_i^+(u)_\mu$ have poles in Π, $\forall \mu$ s.t. $ht(\lambda-\mu) \le k$ and $x_i^-(u)_\mu$ has poles in Π $\forall \mu$ s.t. $ht(\lambda-\mu) < k$.

We will prove $S(k)$ by induction on k.



$S(0)$ is clear since $x_i^+(u)\, V[\lambda] = 0$

$$\xi_i(u)_\lambda = \frac{P_i(u+d_i\hbar)}{P_i(u)}$$

and zeroes of $P_i(u) \subset \Pi$.

Assume $S(k')$ for every $k' \leq k$, where $k \geq 0$. Let us prove $S(k+1)$. Take ν to be a wt. of V s.t. $ht(\lambda - \nu) = k+1$.

- use (Y5)

$$x_i^+(u)_\nu \; x_j^-(v)_{\nu+\alpha_j} = x_j^-(v)_{\nu+\alpha_i+\alpha_j} \, x_i^+(u)_{\nu+\alpha_j} + \frac{\delta_{ij}\hbar}{u-v}\left(\xi_i(v)_{\nu+\alpha_i} - \xi_i(u)_{\nu+\alpha_i}\right)$$

RHS has poles ∉ in $\Pi \times \Pi$. (induction hypothesis).

Assume $x_j^-(v)_{\nu+\alpha_j}$ has a pole at $z \notin \Pi$ of order n. Multiply by $(v-z)^n$ and let $v = z$ to get

$$x_i^+(u)_\nu \left[(v-z)^n \, x_j^-(v)_{\nu+\alpha_j}\Big|_{v=z} \right] = 0$$

i.e. Image of $(v-z)^n x_j^-(v)_{\nu+\alpha_j}\Big|_{v=z}$ is annihilated by all $x_{i,r}^+$. Hence it must be 0, by property (P2). Contradicts the fact that n was order of the pole.

Similarly if $x_i^+(u)_\nu$ has a pole of order n at $z \notin \Pi$ we get

$$\left((u-z)^n \, x_i^+(u)_\nu \Big|_{u=z} \right) x_j^-(v)_{\nu+\alpha_j} = 0$$

$$\underset{X}{\parallel}$$

i.e. X vanishes on the image of all $x_{j,r}^- \, V[\nu+\alpha_j]$. But this image is $V[\nu]$ by (P1). So $X = 0$.

- use (Y23)

$$\xi_i(u)_\nu \, x_j^-(v)_{\nu+\alpha_j} = \frac{u-v-a}{u-v+a} \, x_j^-(v)_{\nu+\alpha_j} \, \xi_i(u)_{\nu+\alpha_j} + \frac{2a}{u-v+a} \, x_j^-(u+a)_{\nu+\alpha_j} \, \xi_i(u)_{\nu+\alpha_j}$$

$\Rightarrow$ on the image of $x_j^-(v)_{\nu+\alpha_j}$, poles of $\xi_i(u)$ are contained in those of $\xi_i(u)_{\nu+\alpha_j}$ or $x_j^-(u+a)_{\nu+\alpha_j}$, hence contained in Π.

(18.5) Let $0 \to V_1 \to V \to V_2 \to 0$ be a short exact seq. If $\sigma(V_1)$, $\sigma(V_2) \subset \Pi$ then $\sigma(V) \subset \Pi$.

Enough to prove it for sl_2. (by fixing index $i \in I$).

Write $V = V_1 \oplus V_2$ as vector space and every $y \in Y$ has the form

$$y = \begin{bmatrix} y^{11} & y^{12} \\ 0 & y^{22} \end{bmatrix}.$$

We know that $\xi(u)^{kk}$, $x^{\pm}(u)^{kk}$ have poles in Π. $(k = 1, 2)$.

(T.S.) $\xi(u)^{12}$ $x^{\pm}(u)^{12}$ have poles in Π.

Assume the contrary. Choose z outside of Π where $\xi(u)^{12}$, $x^{\pm}(u)^{12}$ have poles max'l s.t. these functions are regular at $z + r\hbar$ $(r > 0)$.

Let N = max. of the orders of ~~$\xi(u)^{12}$~~ pole of $\xi(u)^{12}$, $x^{\pm}(u)^{12}$ at $u = z$.

$$H := (u-z)^N \xi(u)^{12}\Big|_{u=z} \qquad X^{\pm} := (u-z)^N x^{\pm}(u)^{12}\Big|_{u=z}.$$

- Use (Y5) and its (1,2) entry

$$(u-v)\left(x^+(u)^{11}x^-(v)^{12} + x^+(u)^{12}x^-(v)^{22} - x^-(v)^{11}x^+(u)^{12} - x^-(v)^{12}x^+(u)^{22}\right) = \hbar\left(\xi(v)^{12} - \xi(u)^{12}\right)$$

$\Longrightarrow$ (multiply by $(u-z)^N$ and let $u \to z$)

$$(z-v)\left(X^+ x^-(v)^{22} - x^-(v)^{11} X^+\right) = -\hbar H$$

set $v = z$ to get $H = 0$.

- use (Y23) for $+$: ~~ξ~~ $(u-v-\hbar)\,\xi(u)\,x^+(v) - (u-v+\hbar)\,x^+(v)\,\xi(u) = -2\hbar\, x^+(u-\hbar)\,\xi(u)$

  Take 1,2 entry. Multiply by $(u-\hbar-z)^N$ and let $u \to z+\hbar$. Using assumptions on z and $H = 0$ we get

$$X^+ \xi(z+\hbar)^{22} = 0.$$

By $\xi(u)^{22}$ is inv. outside of Π. $\Rightarrow X^+ = 0$

- Use (Y23) for $-$: $(u-v+\hbar)\,\xi(u)\,x^-(v) - (u-v-\hbar)\,x^-(v)\,\xi(u) = 2\hbar\,\xi(u)\,x^-(u-\hbar)$

  ... $\Rightarrow X^- = 0$. Contradicts the fact that N was order of the pole of one of $\xi(u)^{12}$, $X^{\pm}(u)^{12}$ $\square$